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On Regular *n*th Root Asymptotic Behavior of Orthonormal Polynomials

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Let μ be a positive measure with compact support on $\mathbf{R}$. We consider the nth root asymptotic behavior of orthonormal polynomials associated with the measure μ. The main result consists of two theorems: (i) a characterization and (ii) a localization theorem. In the first theorem regular nth root asymptotic behavior on a subset of the support of the measure μ is compared with the asymptotic behavior of other polynomial sequences, and equivalences between the different types of behavior are proved. In the second theorem the asymptotic behavior of the original orthonormal polynomials is characterized by the asymptotic behavior of polynomials orthonormal with respect to restrictions of the measure μ.

1. INTRODUCTION

Let μ be a positive measure with compact support $S(\mu) \subseteq \mathbf{R}$, $S(\mu)$ is assumed to be an infinite set, and let

$$P_n(z) = P_n(\mu; z) = \gamma_n z^n + \cdots \qquad (\gamma_n = \gamma_n(\mu) > 0) \tag{1.1}$$

be the *orthonormal polynomial* of degree $n \in \mathbf{N}$ associated with μ; i.e.,

$$\int P_m P_n \, d\mu = \delta_{mn} \qquad \text{for} \quad m, n \in \mathbf{N}, \tag{1.2}$$

where δ_{mn} denotes Kronecker's symbol. Since $S(\mu)$ is an infinite set, all elements of $\{1, x, x^2, ...\}$ are linearly independent in $L^2(\mu)$, and therefore all polynomials (1.1) are uniquely determined by (1.2) and the last assumption in (1.1). We call μ a *weight measure.*

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In this paper we investigate regular nth root asymptotic behavior of the sequence $\{P_n(\mu; z);\ n \in \mathbf{N}\}$; i.e., we study the asymptotic behavior of the sequence

$$\{|P_n(\mu; z)|^{1/n};\ n \in \mathbf{N}\} \qquad \text{as} \quad n \to \infty. \tag{1.3}$$

The statement of the main results requires some preparations. For any weight measure μ the orthonormal polynomials $P_n(\mu; z)$ and their leading coefficients $\gamma_n(\mu)$, $n \in \mathbf{N}$, satisfy certain asymptotic inequalities which are put together in the next lemma:

LEMMA 1.1 (see Section 3.9 of [U12]). *We have*

$$\liminf_{n \to \infty} |P_n(\mu; z)|^{1/n} \geqslant e^{g_\Omega(z, \infty)} \tag{1.4}$$

localy uniformly for $z \in \mathbf{C} \backslash I(\mu)$,

$$\liminf_{n \to \infty} \gamma_n(\mu)^{1/n} \geqslant \frac{1}{\operatorname{cap}(S(\mu))}, \tag{1.5}$$

and for any infinite subsequence $N \subseteq \mathbf{N}$ *we have*

$$\limsup_{n \to \infty, n \in N} |P_n(\mu; z)|^{1/n} \geqslant 1 \tag{1.6}$$

for z quasi everywhere on $S(\mu)$.

In Lemma 1.1 $I(\mu) \subseteq \mathbf{R}$, Ω, and $g_\Omega(z, w)$, $z \in \mathbf{C}^-$, $w \in \Omega$, denote the smallest interval containing the support $S(\mu)$, the domain $\mathbf{C}^- \backslash S(\mu)$, and the (generalized) *Green function* of the dimain Ω, respectively. The Green function $g_\Omega(\cdot, w)$ is harmonic in $\Omega \backslash \{w\}$, subharmonic in $\mathbf{C}$, has a logarithmic pole with residuum 1 at $z = w$, and is equal to 0 quasi everywhere on $\mathbf{C}^- \backslash \Omega$ (see Theorem 2.6 and Chapter IV, No. 2 of [La]). A property is said to hold true *quasi everywhere* (short: *qu.e.*) on a set $S \subseteq \mathbf{C}^-$ if it holds true for every $x \in S$ with possible exceptions on a subset of outer capacity zero. By capacity we mean the *logarithmic capacity* and denote it by $\operatorname{cap}(\cdot)$. If for a domain $\Omega \subseteq \mathbf{C}^-$ with $\infty \in \Omega$, we have $\operatorname{cap}(\mathbf{C}^- \backslash \Omega) = 0$; then we define $g_\Omega(z, w) \equiv \infty$, which is compatible with the defining properties of the Green function.

If equality holds in (1.4), (1.5), and (1.6), then this case is called regular (nth root) asymptotic behavior. Actually, it turns out that it is enough to have equality in only one of the three asymptotic estimates, then equality necessarily follows in the two others. A more precise formulation is given in

LEMMA 1.2 (see Theorem 1 of [U12]). *The following three assertions are equivalent.*

(i) *We have*

$$\lim_{n\to\infty} |P_n(\mu; z)|^{1/n} = e^{g_\Omega(z,\infty)} \tag{1.7}$$

locally uniformly for $z \in \mathbf{C} \setminus I(\mu)$.

(ii) *We have*

$$\lim_{n\to\infty} \gamma_n(\mu)^{1/n} = \frac{1}{\operatorname{cap}(S(\mu))}. \tag{1.8}$$

(iii) *For any infinite subsequence* $N \subseteq \mathbf{N}$ *we have*

$$\limsup_{n\to\infty, n\in N} |P_n(\mu; z)|^{1/n} = 1 \tag{1.9}$$

for z quasi everywhere on $S(\mu)$.

DEFINITION 1.3. The sequence $\{P_n(\mu; z); n \in \mathbf{N}\}$, is said to possess *regular* (*nth root*) *asymptotic behavior* if one of the three assertions of Lemma 1.2 holds true.

Remarks. (1) The case of weight measure μ with $\operatorname{cap}(S(\mu)) = 0$ is not excluded in Lemmas 1.1, 1.2, or Definition 1.3. However, with respect to regular asymptotic behavior, this case is in a certain sense trivial, since the assertions (i), (ii), and (iii) hold true for any weight measures μ if only $\operatorname{cap}(S(\mu)) = 0$, and therefore we always have regular asymptotic behavior for such weight measures.

(2) For monic orthogonal polynomials there exists a definition of regular (nth root) asymptotic behavior that is similar to that in Lemma 1.2. However, since there is no leading coefficient $\gamma_n(\mu)$, $n \in \mathbf{N}$, its role is taken over by the $L^2(\mu)$-norm of the monic orthogonal polynomials (see Section 3.3 of [U12]). In the present paper we consider only orthonormal polynomials. All results can be transferred from one to the other case without difficulties.

The classical, normalized Jacobi polynomials $P_n^{(\alpha,\beta)}$, $\alpha, \beta > -1$, for instance, have regular asymptotic behavior (see Theorem 8.1 of [Fr] for a proof of (1.9)). It is also not too difficult to show that if $S(\mu)$ is a real interval, and the measure μ has a density function bounded away from zero everywhere on $S(\mu)$, then orthonormal polynomials $P_n(\mu; z)$, $n \in \mathbf{N}$, have regular asymptotic behavior (see [Fa] for perhaps the first proof of a result in this direction). However, weight measures μ may be much more general, and the polynomials $P_n(\mu; z)$, $n \in \mathbf{N}$, have nevertheless regular asymptotic behavior. (For examples of quite general weight measures μ, which are supported on Julia sets, we refer to [VA] Section 1.4, or to [BGM]).

Major contributions to the development of the theory of regular (nth root) asymptotic behavior have been obtained by Erdős and Turán [ErTu], Erdős and Freud [ErFr], Ullman [Ull-14], Widom [Wi], and Ullman together with coauthors ([UlWy] and [UWZ]. (For recent reviews on orthogonal polynomials and their asymptotic behavior in general we refer to [Ne] and [Lu]).

One of the problems with practical and theoretical importance is the search for general criteria that guarantee regular asymptotic behavior. A desirable criterion would be one that is necessary and sufficient at the same time. Unfortunately, none of the known criteria has this property. But there are different possibilities to characterize regular asymptotic behavior. We will prove such a characterization result in this paper.

The paper was initiated by the investigation of the convergence and divergence of essentially non-diagonal sequences of Padé approximants (see [St]). There is a special interest in the asymptotic behavior of orthonormal polynomials $P_n(\mu; z)$, $n \in \mathbf{N}$, on certain subsets of the support $S(\mu)$. We note that in (1.9) of Lemma 1.2, and therefore also in Definition 1.3, the whole support has to be considered, while we are now interested in local asymptotic behavior.

Although each orthonormal polynomial $P_n(\mu; \cdot)$ is determined by the whole weight measure μ, experience has shown that the asymptotic behavior on a certain subset of $S(\mu)$ depends only on the restriction of the measure μ to this subset. In Theorem 2.1 (the Characterization Theorem) it will be shown that this empirical observation can indeed be proved for a large class of subsets of $S(\mu)$. In the theorem, regular asymptotic behavior on certain subsets is characterized by a comparison with the asmptotic behavior of other sequences of polynomials. One of the main consequence of Theorem 2.1 is a Localization Theorem (Theorem 2.3) for regular asymptotic behavior, which allows the characterization of regular asymptotic behavior by the asymptotic behavior of the orthonormal polynomials with respect to restrictions of the measure μ.

Besides this original impetus for the present investigation, it turned out during the process of writing that Theorem 2.3 also solves a problem posed in [Lu] (see problem (ii), Section 3.9 of [Lu]).

All results in the paper are proved only for weight measures with compact support $S(\mu)$ in $\mathbf{R}$, but they can be generalized to weight measures μ with compact support in $\mathbf{C}$. These more general results wil be contained in a forthcoming paper by V. Totik and the author, where the problem of nth root asymptotic behavior of orthonormal polynomials with respect to weight measures in $\mathbf{C}$ is treated in a unified approach. The results will also cover generalizations of the Lemmas 1.1 and 1.2 and other material.

The outline of the present paper is as follows: In Section 2 the main results are stated and discussed. In Section 3 and 4 the proof of Theorem 2.1 is prepared by some lemmas from potential theory and the theory of orthogonal polynomials. Section 4 contains two lemmas, which are the key pieces of the proof of Theorem 2.1. Then in Section 5 Theorem 2.1 will be proved.

2. Main Results

Let Π_n denote the set of all polynomials of degree less or equal $n \in \mathbf{N}$, $K \subseteq \mathbf{R}$ a compact set, μ_K the restriction $\mu|_K$ of the measure μ to the set K, Ω_K the domain $\mathbf{C}^- \setminus S(\mu_K)$, and I_K the smallest interval containing $S(\mu_K)$. Besides the orthonormal polynomials $P_n(\mu; z)$, $n \in \mathbf{N}$, we also consider polynomials $P_n(\mu_K; z)$ orthonormal with respect to the restricted measure μ_K, and sequences of arbitrary polynomials $U_n \in \Pi_n$, $n \in \mathbf{N}$, of degree at most n, where the only assumption is that these polynomials are not identically zero.

THEOREM 2.1 (Characterization Theorem). *Let $K \subseteq \mathbf{R}$ be a compact set so that the support $S(\mu_K)$ is an infinite set and*

$$\operatorname{cap}(K \cap S(\mu)) = \operatorname{cap}(\mathring{K} \cap S(\mu)). \tag{2.1}$$

Then the following five assertions are equivalent:

(a) *The sequence $\{P_n(\mu_K; \cdot); n \in \mathbf{N}\}$ has regular (nth root) asymptotic behavior.*

(b) *For any sequence of points $\{z_n\}$ with $z_n \to z_0 \in \mathbf{C}$ as $n \to \infty$, we have*

$$\limsup_{n \to \infty} |P_n(\mu; z_n)|^{1/n} \leqslant e^{g_{\Omega_K}(z_0, \infty)}. \tag{2.2}$$

(c) *For any infinite subsequence $N \subseteq \mathbf{N}$ we have*

$$\limsup_{n \to \infty, n \in N} |P_n(\mu; z)|^{1/n} = 1 \tag{2.3}$$

for z quasi everywhere on $S(\mu_K)$.

(d) *For any infinite sequence of polynomials* $\{U_n \in \Pi_n,\ U_n$ *not identically zero*, $n \in N \subseteq \mathbf{N}\}$ *and for any sequence of points* $\{z_n\}$ *with* $z_n \to z_0 \in \mathbf{C}$ *as* $n \to \infty$, *we have*

$$\limsup_{n \to \infty, n \in N} \left| \frac{U_n(z_n)}{\|U_n\|_{L^2(\mu_K)}} \right|^{1/n} \leqslant e^{g_{\Omega_K}(z_0, \infty)}. \tag{2.4}$$

(e) *For any infinite sequence of polynomials* $\{U_n\}$ *as in assertion* (d) *we have*

$$\limsup_{n \to \infty, n \in N} \left| \frac{U_n(z)}{\|U_n\|_{L^2(\mu_K)}} \right|^{1/n} \leqslant 1 \tag{2.5}$$

for z *quasi everywhere on* $S(\mu_K)$.

If $S(\mu_K)$ *is a regular set with respect to the solution of the Dirichlet problem in the domain* Ω_K, *then in assertions* (b) *and* (d) *the asymptotic inequalities* (2.2) *and* (2.4) *hold locally uniformly in* $z \in \mathbf{C}$, *in assertion* (e) *the asymptotic inequality* (2.5) *holds not only quasi everywhere, but uniformly on* $S(\mu_K)$, *and in* (2.3) *of assertion* (c) *we have an upper inequality* "$\leqslant$" *uniformly on* $S(\mu_K)$ *in addition to the equality quasi everywhere stated in* (2.3).

Remarks. (1) It is easy to see that the formulations given in assertions (b) and (d) imply that the asymptotic inequalities (2.2) and (2.4) hold locally uniformly in every open set in which $g_{\Omega_K}(z, \infty)$ is continuous. This is, for instance, always the case in Ω_K.

(2) The assumption that $S(\mu_K)$ is an infinite set is necessary in order that all polynomials $P_n(\mu_K; z)$, $n \in \mathbf{N}$, are defined uniquely.

If $\operatorname{cap}(S(\mu_K)) = 0$, then it can easily be verified that assumption (2.1) and all five assertions of Theorem 2.1 hold true independently of any other property of the weight measure μ (compare Remark 1 to Lemma 1.2), and therefore in this special case the five assertions of Theorem 2.1 are trivially equivalent.

(3) By example 2.4 below it will be shown that assumption (2.1) cannot be dropped without replacement. If we drop assumption (2.1), then only the implication

$$((\mathrm{a}) \vee (\mathrm{d}) \vee (\mathrm{e})) \Rightarrow ((\mathrm{b}) \wedge (\mathrm{c})) \tag{2.6}$$

can be proved. The assertions in both groups of (2.6) remain equivalent.

(4) The two inclusions

$$\overline{\mathring{K} \cap S(\mu)} \subseteq S(\mu_K) \subseteq K \cap S(\mu) \tag{2.7}$$

can easily be verified. With assumption (2.1) they imply that

$$\operatorname{cap}(\mathring{K} \cap S(\mu)) = \operatorname{cap}(S(\mu_K)) = \operatorname{cap}(K \cap S(\mu)), \tag{2.8}$$

and

$$g_{\mathbf{C} \setminus (K \cap S(\mu))}(z, \infty) \equiv g_{\Omega_K}(z, \infty). \tag{2.9}$$

The last identity follows from the uniqueness of the Green function (see Theorem 2.6 and Chapter VI, No. 2 of [La]).

From (2.7) and (2.8) it also follows that assumption (2.1) implies that the two sets $K \cap S(\mu)$ and $S(\mu_K)$ can differ at most in a set of capacity zero. Therefore, we can replace $S(\mu_K)$ by $K \cap S(\mu)$ in the assertions (c) and (e).

(5) If one chooses $K = I(\mu)$, then assumption (2.1) is satisfied and we have $\mu = \mu_K$ and $\Omega = \Omega_K$. This case is not trivial and it is interesting, since then the equivalence of the assertions (a), (d), and (e) gives a characterization of regular asymptotic behavior of the orthonormal polynomials $P_n(\mu; \cdot)$, $n \in N$, by the asymptotic behavior of other sequences of polynomials U_n.

(6) Since the right-hand side of (2.4) is equally 1 quasi everywhere on $S(\mu_K)$ (see Theorem 2.6′ of [La]), assertion (e) is a special case of assertion (d).

Theorem 2.1 will be proved only in Section 5 after preparations in the Sections 3 and 4.

Using Lemma 1.1 we can deduce upper and lower asymptotic bounds for the nth root of the orthonormal polynomials $P_n(\mu; \cdot)$, $n \in \mathbf{N}$, from the assertions (a) and (b). They are stated in the next corollary. We note that the bounds (2.10) refer to the original orthonormal polynomials $P_n(\mu; z)$, while the assumption in the corollary is related to the asymptotic behavior of the orthonormal polynomials $P_n(\mu_K; z)$ associated with the restricted weight measure μ_K.

COROLLARY 2.2. *Let $K \subseteq \mathbf{R}$ be a compact set that satisfies the assumptions of Theorem 2.1, and let us assmume that the sequence $\{P_n(\mu_K; \,)); n \in \mathbf{N}\}$ has regular asymptotic behavior. Then we have*

$$e^{g_\Omega(z, \infty)} \leqslant \liminf_{n \to \infty} |P_n(\mu; z)|^{1/n} \leqslant \limsup_{n \to \infty} |P_n(\mu; z)|^{1/n} \leqslant e^{g_{\Omega_K}(z, \infty)}, \tag{2.10}$$

where the first inequality in (2.10) holds true locally uniformly for $z \in \mathbf{C} \setminus I(\mu)$, the last inequality holds for all $z \in \mathbf{C}$, and it holds locally uniformly in every open set in which $g_{\Omega_K}(z, \infty)$ is continuous.

From the equivalence of the assertions (a) and (c) in Theorem 2.1 we deduce a localization theorem, which is the second main result of the paper.

THEOREM 2.3 (Localization Theorem). *Let* $J_j := [a_j, b_j]$, $a_j < b_j$, $j = 1, 2, \ldots$, *be a countable collection of intervals, which cover the support* $S(\mu)$. *Then the sequence* $\{P_n(\mu; \cdot); n \in \mathbf{N}\}$ *has regular (nth root) asymptotic behavior if, and only if, all sequences* $\{P_n(\mu_{J_j}; \cdot); n \in \mathbf{N}\}$, $j = 1, 2, \ldots$ *with* $\operatorname{cap}(J_j \cap S(\mu)) > 0$ *have regular (nth root) asymptotic behavior.*

Proof (of Theorem 2.3). It is easy to see that for each set $K := J_j$, $j = 1, 2, \ldots$, the assumption (2.1) of Theorem 2.1 is satisfied. The restriction of the measure μ of J_j is denoted by μ_{J_j}, $j = 1, 2, \ldots$.

We assume that $\operatorname{cap}(J_j \cap S(\mu)) > 0$ for $j = 1, \ldots, m_0$ ($m_0 \in N \cup \{\infty\}$), and $\operatorname{cap}(J_j \cap S(\mu)) = 0$ for $j > m_0$. Let B be the union of the sets $S(\mu_{J_j})$, $j = 1, \ldots, m_0$. Because of (2.8) we have $\operatorname{cap}((J_j \cap S(\mu)) \setminus S(\mu_{J_j})) = 0$ for all $j = 1, 2, \ldots$, and the set $S(\mu) \setminus B$ is also of capacity zero since the union of countably many sets of capacity zero is again a set of capacity zero (see the corollary to Theorem 2.2 of [La]).

Let us assume that the orthonormal polynomials $P_n(\mu_{J_j}; \cdot)$, $j = 1, \ldots, m_0$, have regular asymptotic behavior for $n \to \infty$. Since for each $K := J_j$, $j = 1, \ldots, m_0$, the assumption (2.1) of Theorem 2.1 is satisfied, we know from assertion (c) of Theorem 2.1 that equality (2.3) holds true for z quasi everywhere on $S(\mu_{J_j})$ for $j = 1, \ldots, m_0$. This implies that (2.3) is proved for z quasi everywhere on B, and therefore also for z quasi everywhere on $S(\mu)$. By assertion (iii) of Lemma 1.2 we then know that the sequence $\{P_n(\mu; \cdot); n \in \mathbf{N}\}$ has regular asymptotic behavior.

Let us now assume that the sequence $\{P_n(\mu; \cdot); n \in \mathbf{N}\}$ has regular asymptotic behavior. By Definition 1.3 and assertion (iii) of Lemma 1.2 this implies that (2.3) holds true for z quasi everywhere on $S(\mu)$. From the equivalence of assertion (a) and (c) of Theorem 2.1 it then immediately follows that each sequence $\{P_n(\mu_{J_j}; \cdot); n \in \mathbf{N}\}$, $j = 1, \ldots, m_0$, has regular asymptotic behavior. Q.E.D.

The next example shows that assumption (2.1) of Theorem 2.1 is necessary in one or the other form.

EXAMPLE 2.4. Let C be the classical Cantor set on $[0, 1]$, and let μ_2 be a probability measure with $S(\mu_2) = C$ so that the orthonormal polyomials $P_n(\mu_2; \cdot)$, $n \in \mathbf{N}$, do not have regular asymptotic behavior. The existence of such a measure μ_2 follows from [U12; Theorem 2], but it is also not difficult to describe a construction. For instance, the measure

$$\mu_2 := \sum_{n=1}^{\infty} \alpha^{n^2} \delta_{x_n} \Big/ \sum_{n=1}^{\infty} \alpha^{n^2} \qquad (\delta_x \text{ denotes Dirac's measure}) \qquad (2.11)$$

has the required property if $\{x_n\} \subseteq C$ is a sequence of points dense in C and $0 < \alpha < 1$.

We now consider the weight measure $\mu := \mu_1 + \mu_2$, where μ_1 is the linear Lebesgue measure on [0, 1]. It can easily be verified, for instance, by the Erdős–Turán criterion (see Section 3.4 of [U12]), that the sequence $\{P_n(\mu; \cdot); n \in \mathbf{N}\}$ has regular asymptotic behavior. Hence, assertion (b) of Theorem 2.1 holds true. On the other hand, we have $P_n(\mu_C; \cdot) = P_n(\mu_2; \cdot)$ for all $n \in \mathbf{N}$ since the linear Lebesgue measure of C is equally zero. This implies that assertion (a) of Theorem 2.1 is false if we take $K = C$. The Cantor set C is of positive capacity and has no inner points. Hence, assumption (2.1) is not satisfied.

3. Notations and Some Lemmas

We assemble some lemmas from potential theory and the theory of orthogonal polynomials. Only the last four of these lemmas cannot be found elswhere and have to be proved here. Two limit functions L_2 and L_0, which will be introduced in Definition 3.3, are basic for the results of Section 4 and the proof of Theorem 2.1.

By $Z(P)$ we denote the *set of zeros* of a polynomial P, taking account of multiplicities. Thus, $\deg(P) = \operatorname{card} Z(P)$. For a finite set $Z \subseteq \mathbf{C}$ of n numbers, the monic polynomial

$$Q(Z; z) := \prod_{w \in Z} (z - w) = z^n + \cdots \in \Pi_n, \qquad Q(\phi; \cdot) \equiv 1, \tag{3.1}$$

is denoted by $Q(Z; \cdot)$. The counting measure of a finite set Z is denoted by ν_Z, and for a polynomial P the measure $\nu_{Z(P)}$ is denoted by ν_P. This last measure is called zero distribution of the polynomial P. Thus, we have $\nu_Z = \nu_{Q(Z; \cdot)}$.

As (*logarithmic*) *potential* of a measure μ in $\mathbf{C}$, we define

$$q(\mu; z) := \int \log |z - w| \, d\mu(w) \qquad \text{for} \quad z \in \mathbf{C}. \tag{3.2}$$

This differs by a negative sign from the more usual definition (see Chapter II Section 4 of [La]), but we prefer (3.2) because of its close connection with monic polynomials. For any monic polynomial Q we have $q(\nu_Q; z) = \log |Q(Z)|$.

We say that a sequence of measures $\{\mu_n\}$ *converges weakly* to a measure μ, written as $\mu_n \xrightarrow{*} \mu$, if for any function f continuous on the Riemann sphere $\bar{\mathbf{C}}$, we have $\int f \, d\mu_n \to \int f \, d\mu$ as $n \to \infty$. Since the unit ball B of positive measures (with respect to the norm $\|\cdot\|$ of total variation) is weakly compact, from every infinite sequence of measures $\{\mu_n\} \subseteq B$ we can select an infinite subsequence that is weakly convergent. This result is often called *Helly's Selection Theorem.*

The next two lemmas contain some basic results or immediate consequences of basic results from potential theory.

LEMMA 3.1 (see Theorem 3.8 of [La]). *Let ν be a probability measure with compact support $S(\nu) \subseteq \mathbf{C}$. Then there exists a sequence of finite sets $Z_n \subseteq S(\nu)$, $n \in \mathbf{N}$, each set contains n points, and*

$$\nu_n := \frac{1}{n} \nu_{Z_n} \xrightarrow{*} \nu \qquad as \quad n \to \infty. \tag{3.3}$$

From (3.3) *it follows that*

$$\limsup_{n \to \infty} q(\nu_n; z) = q(\nu; z) \qquad for \quad z \text{ qu.e on } S(\nu), \tag{3.4a}$$

$$\lim_{n \to \infty} q(\nu_n; z) = q(\nu; z) \qquad \text{locally uniformly for} \quad z \in \mathbf{C} \setminus S(\nu), \tag{3.4b}$$

and for any sequence of points $\{z_n\}$ with $z_n \to z_0 \in \mathbf{C}$ as $n \to \infty$, we have

$$\limsup_{n \to \infty} q(\nu_n; z_n) \leqslant q(\nu; z_0). \tag{3.4c}$$

The limit (3.4a) follows from (3.3) and the *lower envelope theorem* of potential theory (see Theorem 3.8 of [La]), the limit (3.4b) directly follows from (3.3), and the limit (3.4c) follows from the *principle of descent* in potential theory (see Theorem 1.3 of [La]).

LEMMA 3.2 (see Chapter IV Section 1 of [La]). *Let ν be a positive measure with compact support in $\mathbf{C}$, $K \subseteq \mathbf{R}$ a compact set of positive capacity that does not separate $\bar{\mathbf{C}}$, and $\omega_w = \omega_{w,K}$ the harmonic measure representing the point $w \in \bar{\mathbf{C}} \setminus K$ on K. Then there exists a measure $\hat{\nu}$ on K with the property that*

$$q(\nu; z) = q(\hat{\nu}; z) + c \qquad for \quad z \text{ qu.e. on } K, \tag{3.5}$$

and we have

$$\hat{\nu} = \int_{\mathbf{C} \setminus K} \omega_w \, d\nu(w), \qquad \|\nu\| = \|\hat{\nu}\|, \tag{3.6a}$$

and

$$c = \int_{\mathbf{C} \setminus K} g_{\mathbf{C} \setminus K}(w, \infty) \, d\nu(w). \tag{3.6b}$$

If the set K is regular (with respect to the Dirichlet problem in $\bar{\mathbf{C}}\setminus K$), then in (3.5) equality holds true for all $z \in K$.

The measure $\hat{\nu}$ is called *balayage measure* of ν, and the process of going from $q(\nu; z)$ to $q(\hat{\nu}; z)+c$ is called *balayage* or sweeping out of the measure ν from $\bar{\mathbf{C}}\setminus K$ onto K (see Chapter IV Section 1 of [La]). The non-negative constant c appears in (3.5) since the domain $\bar{\mathbf{C}}\setminus K$ is unbounded (see Corollary 3 to Theorem 4.2 of [La]).

We next introduce two limit functions, which are of basic importance in the proof of Theorem 2.1. Their properties will be studied in the subsequent lemmas.

DEFINITION 3.3. *Let μ be a positive measure with compact support on $\mathbf{R}$, $N \subseteq \mathbf{N}$ an infinite subsequence, and $\{U_n\} = \{U_n = z^m + \cdots \in \Pi_n;\ m \leqslant n \in N\}$ a sequence of monic polynomials. We define the upper (logarithmic) limit function L_2 by*

$$\begin{aligned} \tilde{L}_2(z) &:= \tilde{L}_2(\mu, \{U_n\}; z) := \limsup_{n\to\infty,\, n\in N} \frac{1}{n}\left[q(\nu_{U_n}; z) - \log \|U_n\|_{L^2(\mu)}\right], \\ L_2(z) &:= L_2(\mu; \{U_n\}; z) := \limsup_{w\to z} \tilde{L}_2(w). \end{aligned} \tag{3.7a}$$

Of special interest is the case of sequences of monic orthogonal polynomials $\{\gamma_n(\mu)^{-1} P_n(\mu;\cdot);\ n \in N \subseteq \mathbf{N}\}$, for which we introduce a separate notation. We define

$$L_0(z) := L_0(\mu, N; z) := L_2\left(\mu, \left\{\frac{1}{\gamma_n(\mu)} P_n(\mu;\cdot); n \in N\right\}; z\right), \tag{3.7b}$$

and call function (3.7b) the *upper limit function (for orlthonormal polynomials).*

Remarks. (1) The right-hand side of (3.7a) can be rewritten by using the identity

$$\log \left|\frac{U_n(z)}{\|U_n\|_{L^2(\mu)}}\right|^{1/n} \equiv \frac{1}{n}\left[q(\nu_{U_n}; z) - \log \|U_n\|_{L^2(\mu)}\right], \tag{3.8}$$

which shows that L_2 and L_0 are defined by the nth root of the modulus of the polynomials normalized in $L^2(\mu)$. The left-hand side of (3.8) further shows that the polynomials U_n do not necessarily have to be monic, as has been assumed in Definition 3.3. The normalization can always be achieved by a multiplication with a non-zero constant, and such a multiplication will not change the value of (3.8) and therefore lets L_2 be invariant.

(2) The functions L_0 and L_2 may be identical ∞. It is not difficult to verify that if this is the case at a finite point $z \in \mathbf{C}$, then the function L_0 or L_2 is identically ∞ everywhere in $\mathbf{C}$.

(3) Like in the proof of the principle of descent in Theorem 1.7 of [La], or by considering weakly convergent subsequences of $\{\nu_{U_n}; n \in N\}$ and then applying the principle of descent, it can be shown that for any sequence of points $\{z_n; n \in N\}$ with $z_n \to z_0 \in \mathbf{C}$ as $n \to \infty$, $n \in N$, we have

$$\limsup_{n \to \infty,\, n \in N} L_2(\mu, \{U_n\}; z_n) \leqslant L_2(\mu, \{U_n\}; z_0), \tag{3.9}$$

and for every $z_0 \in \mathbf{C}$ there exists a sequence $\{z_n\}$ such that we have equality in (3.9).

If we know that the zero distributions ν_{U_n} and the $L^2(\mu)$-norms of the polynomials U_n converge, then the lower envelope theorem of potential theory (see Theorem 3.8 of [La]) gives us a representation of the limit function L_2 as a logarithmic potential plus a constant. This is the constant c_1 in the next lemma.

LEMMA 3.4 (see Remark 2 to Theorem 3.8 of [La]). *Let $\{U_n = z^m + \cdots \in \Pi_n;\ m \leqslant n \in N \subseteq \mathbf{N}\}$ be an infinite sequence of monic polynomials, with all its zeros contained in a compact set $V \subseteq \mathbf{C}$, and let*

$$\frac{1}{n}\nu_{U_n} \xrightarrow{*} \nu_1 \qquad \text{and} \qquad \frac{1}{n}\log \|U_n\|_{L^2(\mu)} \to c_1 \in \mathbf{R} \cup \{-\infty\}$$

$$\text{as} \quad n \to \infty, \quad n \in N, \tag{3.10}$$

hold true. Then we have

$$L_2(\mu, \{U_n\}; z) \equiv q(\nu_1; z) - c_1, \tag{3.11}$$

and ν_1 is a non-negative measure with $S(\nu_1) \subseteq V$.

We note that in (3.10) the case $c_1 = \infty$ has been excluded. The support of ν_1 may be unbounded and therefore $q(\nu_1; z)$ may be identical infinity. The measure ν_1 is not necessarily a probability measure; it may even be identical zero.

In the next two lemmas we prove more specific properties of the limit function L_2.

LEMMA 3.5. *Let $K \subseteq \mathbf{R}$ be a compact set with $\operatorname{cap}(S(\mu_K)) > 0$, and $\{U_n = z^m + \cdots \in \Pi_n;\ m \leqslant n \in N \subseteq \mathbf{N}\}$ an infinite sequence of monic polynomials. If*

$$\tilde{L}_2(\mu, \{U_n\}; z) \leqslant g_{\Omega_K}(z, \infty) \qquad \text{for all} \quad z \in \mathbf{C} \setminus I_K, \tag{3.12}$$

or if

$$\tilde{L}_2(\mu, \{U_n\}; z) \leqslant 0 \qquad \textit{for} \quad z \textit{ qu.e. on } S(\mu_k), \tag{3.13}$$

then

$$L_2(\mu, \{U_n\}; z) \leqslant g_{\Omega_K}(z, \infty) \qquad \textit{for all} \quad z \in \mathbf{C}. \tag{3.14}$$

Remark. If in (3.12) and (3.13) the function $\tilde{L}_2$ is replaced by its upper regularization, the function L_2, then the assumptions of the lemma are strengthened, and conclusion (3.14) remains true.

Proof. Basically, the inequality (3.14) follows from (3.12) because of the continuity of logarithmic potentials in Cartan's fine topology (for a definition see Section 3 of Chapter V of [La]), and (3.14) follows from (3.13) as a consequence of the principle of domination (see Theorem 1.27) of [La]). We will give the proof in more detail:

Because of $\operatorname{cap}(S(\mu_K)) > 0$, the Green function $g(z, \infty) = g_{\Omega_K}(z, \infty)$ can be represented as a logarithmic potential plus a constant (see Theorem 2.6 and Chapter IV, No. 3 of [La]), and therefore it is subharmonic and especially upper semicontinuous in $\mathbf{C}$. This implies that if (3.12) holds true then it also holds true with $\tilde{L}_2$ replaced by L_2. We will now use this stronger assumption.

The lemma will be proved first under the additional assumption that the sequence $\{U_n\}$ of polynomials has the properties assumed in Lemma 3.4; i.e., the zeros of all U_n are contained in a compact set $V \subseteq \mathbf{C}$, and the two limits in (3.10) exist. We then know from (3.11) that the limit function L_2 is the logarithmic potential plus a constant.

Let us assume that (3.12) holds true. Both functions, $g_{\Omega_K}(z, \infty)$ and L_2, are logarithmic potentials plus a constant, and therefore they are continuous in Cartan's fine topology; and further Ω_K is dense in $\mathbf{C}$ in this topology. Hence, the inequality (3.12), with $\tilde{L}_2$ replaced by L_2, extends to the whole complex plane $\mathbf{C}$, which proves (3.14).

Let us now assume that (3.13) holds true. From the lower envelope theorem (see Theorem 3.8 of [La]) together with Lemma 3.4 it follows that $\tilde{L}_2$ and L_2 are equal quasi everywhere. Hence, (3.13) remains true if $\tilde{L}_2$ is replaced by L_2. Both functions, $g_{\Omega_K}(z, \infty)$ and L_2, are logarithmic potentials plus a constant. Since $\operatorname{cap}(S(\mu_K)) > 0$, the logarithmic potential representing $g_{\Omega_K}(z, \infty)$ is generated by a measure of finite energy, which is contained in $S(\mu_K)$ (see Theorem 2.6, Chapter IV, No. 3, and Chapter I Section 4 of [La]). The inequality (3.14) therefore follows from (3.13) by the principle of domination (see Theorem 1.27 of [La]).

We finally show that the additional assumptions are not really necessary. If the sequence $\{U_n\}$ is such that the zeros of all U_n are contained in a compact set $V \subseteq \mathbf{C}$, but the two limits in (3.10) do not exist, then by Helly's Selection Theorem any infinity subsequence of $\{U_n\}$ contains a subsequence such that the two limits in (3.10) exist. With the usual compactness arguments then (3.14) follows from (3.13). We will give more details: Let us assume that (3.14) is wrong. Then there exists $z_0 \in \mathbf{C}$ with

$$L_2(z_0) = L_2(\mu, \{U_n\}; z_0) > g_{\Omega_K}(z_0, \infty). \tag{3.15}$$

From the definition of L_2, it then follows that there exists an infinite subsequence $N_0 \subseteq N$ and a sequence of points $z_n \in \mathbf{C}$, $n \in N_0$, with

$$\frac{1}{n}\left[q(\nu_{U_n}; z_n) - \log \|U_n\|_{L^2(\mu)}\right] \to L_2(z_0) \qquad \text{and} \qquad z_n \to z_0 \tag{3.16}$$

for $n \to \infty$, $n \in N_0$. By Helly's Selection Theorem we can select an infinite subsequence $N_1 \subseteq N_0$ such that the two limits in (3.10) exist. Since for this case (3.14) has already been proved, it follows with (3.16) that

$$L_2(z_0) = L_2(\mu, \{U_n; n \in N_1\}; z_0) \leqslant g_{\Omega_K}(z_0, \infty),$$

which contradicts (3.15).

If there does not exist a compact set $V \subseteq \mathbf{C}$ containing all zeros of all polynomials U_n, then we may choose $R > 0$ with $S(\mu) \subseteq \{|z| \leqslant R\}$ and $k > 1$ arbitrary, and factor each polynomial U_n in the product $V_n W_n$ of two monic polynomials V_n and W_n such that V_n has all its zeros in $\{|z| \leqslant kR\}$ and W_n has all its zeros in the complement. From these definitions it immediately follows that

$$(k-1)R \leqslant \left|\frac{U_n(z)}{V_n(z)}\right|^{1/n} \leqslant (k+1)R \qquad \text{for all} \quad |z| \leqslant R,$$

and an elementary calculation shows that

$$\log\frac{k-1}{k+1} \leqslant |L_2(\mu, \{U_n\}; z) - L_2(\mu, \{V_n\}; z)| \leqslant \log\frac{k+1}{k-1} \tag{3.17}$$

for all $|z| \leqslant R$. For the sequence $\{V_n\}$ the lemma is prove. With (3.17) this proof carries over to the original sequence $\{U_n\}$ since we may choose R and k arbitrarily large. Q.E.D.

LEMMA 3.6. *Let the compact set $K \subseteq \mathbf{R}$ and the sequence $\{U_n\}$ be the same as in Lemma 3.5. If*

$$\operatorname{cap}\{z \in S(\mu_K); L_2(\mu, \{U_n\}; z)\} > 0, \tag{3.18}$$

then there exists an infinite subsequence $N_1 \subseteq N$ so that the two limits in (3.10) *of Lemma* 3.4 *exists for this sequence, and the measure ν_1 and the constant c_1 satisfy*

$$\operatorname{cap}\{z \in S(\mu_K);\ q(\nu_1; z) - c_1 > 0\} > 0 \tag{3.19}$$

or $c_1 = \infty$.

Proof. Let x_0 be a regular point of $S(\mu_K)$. (Regular with respect to the Dirichlet problem in $\bar{\mathbf{C}} \setminus S(\mu_K)$). Since the set of irregular points of a compact set is of capacity zero (see Lemma 5.2 of [La]), assumption (3.18) implies that there exists a regular point $x_0 \in S(\mu_K)$ with

$$L_2(x_0) := L_2(\mu, \{U_n\}; x_0) > 0. \tag{3.20}$$

As in the second part of the proof of Lemma 3.5 it then follows from the definition of L_2 that there exists an infinite subsequence $N_0 \subseteq N$ and points $x_n \in \mathbf{C}$, $n \in N_0$, with

$$\frac{1}{n}\,[q(\nu_{U_n}; x_n) - \log \|U_n\|_{L^2(\mu)}] \to L_2(x_0) \qquad \text{and} \quad x_n \to x_0$$

for $n \to \infty$, $n \in N_0$. By Helly's Selection Theorem we can select an infinite subsequence of $N_1 \subseteq N_0$ such that the two limits in (3.10) exist. If $c_1 < \infty$, then we know from Lemma 3.4 that

$$L_2(\mu, \{U_n; n \in N_1\}; z) \equiv q(\nu_1; z) - c_1. \tag{3.21}$$

The selection of the subsequence of N_1 implies that $L_2(\mu, \{U_n; n \in N_1\}; x_0) \geqslant \tilde{L}_2(\mu, \{U_n; n \in N_1\}; x_0) > 0$.

Let us now assume that (3.19) is false. Then because (3.21), assumption (3.13) of Lemma 3.5 is satisfied for $L_2(\mu, \{U_n; n \in N_1\}; z)$, and from (3.14) it follows that

$$L_2(\mu, \{U_n; n \in N_1\}; z) \leqslant g_{\Omega_K}(x_0, \infty).$$

Since x_0 has been chosen as a regular point of $S(\mu_K)$, the right-hand side of the last inequality must be equally zero (see Lemma 4.5 of [La]), but this contradicts our assumption made in (3.20). Hence, (3.19) is proved. Q.E.D.

While the last three lemmas were concerned with the upper limit functon L_2, we now turn to the limit function L_0, which is associated with subsequences of monic orthogonal polynomials, which will be denoted by

$$Q_n(\mu; z) = \frac{1}{\gamma_n(\mu)}\, P_n(\mu; z) = z^n + \cdots \in \Pi_n, \qquad n \in \mathbf{N}, \tag{3.22}$$

where, as in (3.7b), $\gamma_n(\mu)$ is the leading coefficient of $P_n(\mu; \cdot)$ introduced in (1.1). It is well known that the monic orthogonal polynomials are uniquely defined by the minimality property

$$\|Q_n(\mu; \cdot)\|_{L^2(\mu)} = \min\{\|U\|_{L^2(\mu)};\ U(z) = z^n + \cdots \in \Pi_n\} \tag{3.23}$$

(see Theorem 2.2 of [Fr]).

LEMMA 3.7. (a) *For any infinite subsequence* $N \subseteq \mathbf{N}$, *we have*

$$L_0(\mu, N; z) \geqslant g_\Omega(z, \infty) \qquad \textit{for all} \quad z \in \mathbf{C}. \tag{3.24}$$

(b) *If for an infinite subsequence* $N \subseteq \mathbf{N}$ *the two limits in* (3.10) *exist with* $U_n(z) = Q_n(\mu; z)$, *then* v_1 *is a probability measure with* $S(v_1) \subseteq S(\mu)$. *If* $c_1 > -\infty$, *then the measure* v_1 *is of finite energy*. (For a definition of finite energy see Chapter I Section 4 of [La]).

Proof. (a) The inequality (3.24) is an immediate consequence of (1.4) and (1.6) of Lemma 1.1.

(b) The polynomial $Q_n(\mu; z)$, $n \in \mathbf{N}$, is of exact degree n, all zeros of $Q_n(\mu; z)$ are contained in $I(\mu)$, and every component of $I(\mu) \setminus S(\mu)$ contains at most one zero (see Chapter I of [Fr]). Therefore, v_1 is a probability measure, and $S(v_1) \subseteq S(\mu)$.

From (3.24) and from (3.11) of Lemma 3.4 it follows that in case of $c_1 > -\infty$ the potential $q(v_1; z)$ is bounded from below, which implies that v_1 is of finite energy (see Chapter I, Section 4 of [La]). Q.E.D.

LEMMA 3.8. *The following four assertions are equivalent*:

(a) *The sequence* $\{P_n(\mu; z)\}$ *has regular asymptotic behavior.*

(b) $L_0(\mu, \mathbf{N}; z) \equiv g_\Omega(z, \infty)$.

(c) $\operatorname{cap}\{z \in S(\mu);\ L_0(\mu, \mathbf{N}; z) > 0\} = 0$.

(d) $(1/n) \log \|Q_n(\mu; \cdot)\|_{L^2(\mu)} \to \log \operatorname{cap}(S(\mu))$ *as* $n \to \infty$, $n \in \mathbf{N}$.

Proof. In many respects the lemma is a reformulation of Lemma 1.2, only we now use the limit function L_0. We prove the lemma by the sequence (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (d) $\Rightarrow$ (a).

(a) $\Rightarrow$ (b): The implication follows from assertion (i) of Lemma 1.2 together with Lemma 3.5, where we have to choose $K = I(\mu)$, $N = \mathbf{N}$, and $U_n = P_n(\mu; \cdot)$, $n \in \mathbf{N}$.

(b) $\Rightarrow$ (c): The Green function $g_\Omega(z, \infty) = 0$ quasi everywhere on $S(\mu)$ (see Theorem 2.6′ of [La]).

(c) $\Rightarrow$ (d): Assertion (c) together with Definition 3.3 and (1.6) implies that assertion (iii) of Lemma 1.2 is satisfied. Then assertion (d) follows from assertion (ii) of Lemma 1.2 and the identity

$$\|Q_n(\mu; \cdot)\|_{L^2(\mu)} = \frac{1}{\gamma_n(\mu)}. \tag{3.25}$$

(d) $\Rightarrow$ (a): Regular asymptotic behavior of the sequence $\{P_n(\mu; \cdot)\}$ follows by assertion (ii) of Lemma 1.2 from assertion (d) together with identity (3.25). Q.E.D.

4. Two Basic Lemmas

Two key lemmas for the proof of Theorem 2.1 will be proved. The more difficult one is the second lemma.

As in Section 2, we denote the restriction of the measure μ to a compact set $K \subseteq \mathbf{R}$ by μ_K.

Lemma 4.1. *If*

$$\operatorname{cap}\{z \in S(\mu_K); L_0(\mu, \mathbf{N}; z) > 0\} > 0, \tag{4.1}$$

then we also have

$$\operatorname{cap}\{z \in S(\mu_K); L_0(\mu_K, \mathbf{N}; z) > 0\} > 0. \tag{4.2}$$

Proof. Let us assume that (4.1) is true. We prove that this implies the existence of a sequence $\{U_n(z) = z^n + \cdots \in \Pi_n;\ n \in N \subseteq \mathbf{N}\}$ of monic polynomials with

$$\limsup_{n \to \infty,\ n \in N} \frac{1}{n} \log \|U_n\|_{L^2(\mu_K)} < \log \operatorname{cap}(S(\mu_K)). \tag{4.3}$$

From the minimality property (3.23) and the equivalence of the assertions (c) and (d) of Lemma 3.8, it then follows that (4.2) holds true. In Lemma 3.8 we have to replace μ by μ_K.

The basic idea for the construction of the polynomials U_n consists in a modification of the orthogonal monic polynimials $Q_n(\mu; z)$, $n \in \mathbf{N}$. The zeros of $Q_n(\mu; z)$ are moved from outside of $S(\mu_K)$ onto the set $S(\mu_K)$. Since we do not know whether the set $S(\mu_K)$ is regular, some technical precautions have to be taken.

From (4.1) and Lemma 3.6 we know that there exists an infinite subsequence $N \subseteq \mathbf{N}$ with

$$\frac{1}{n} \nu_{Q_n(\mu;\,\cdot\,)} \xrightarrow{*} \nu_0, \qquad \frac{1}{n} \log \|Q_n(\mu;\,\cdot\,)\|_{L^2(\mu)} \to c_0 \in \mathbf{R} \cup \{-\infty\}$$

$$\text{for} \quad n \to \infty, \quad n \in N, \tag{4.4}$$

and

$$\operatorname{cap}\{z \in S(\mu_K);\, q(\nu_0; z) > c_0\} > 0. \tag{4.5}$$

In part (b) of Lemma 3.7 it has been shown that ν_0 is a probability measure with $S(\nu_0) \subseteq S(\mu)$.

Since the capacity is continuous from the outside, we can find sets $S \subseteq \mathbf{R}$ that consist of finitely many closed intervals, contain $S(\mu_K)$ in its interior, and the capacity $\operatorname{cap}(S)$ approaches $\operatorname{cap}(S(\mu_K))$ as close as we want.

The measure ν_0 is in general not restricted to the set S. We consider the decomposition

$$\nu_0 = \nu_{00} + \nu_{01} \quad \text{with} \quad \nu_{00} := \nu_0|_S \qquad \text{and} \qquad \nu_{01} := \nu_0|_{\mathbf{R}\setminus S}. \tag{4.6}$$

Using the balayage technique, which has been described in Lema 3.2, we sweep the measure ν_{01} out of $\mathbf{C}\setminus S$ onto S. The balayage measure, which is defined by (3.6a), is denoted by ν_{11}, and we define

$$\nu_1 := \nu_{00} + \nu_{11}, \tag{4.7}$$

which is a probability measure on S. From Lemma 3.2 and the assumed regularity of S, we know that

$$q(\nu_1; z) - q(\nu_0; z) = c_1 \qquad \text{for all} \quad z \in S, \tag{4.8}$$

where c_1 is non-positive constant given by

$$c_1 = -\int g_{\bar{\mathbf{C}}\setminus S}(x, \infty)\, d\nu_{11}(x) \tag{4.9}$$

(see (36.b) of Lemma 3.2).

Like the measures ν_1 and ν_{11}, so also the constant c_1 depends on the choice of S. We next show that if S is chosen sufficiently small, then

$$c_0 + c_1 < \log \operatorname{cap}(S(\mu_K)). \tag{4.10}$$

If $c_0 = -\infty$, then nothing remains to be proved. We assume that $c_0 > -\infty$. It follows from (1.4) of Lemma 1.1 by the same arguments as used in Lemma 3.5 that

$$q(\nu_0; z) \geqslant c_0 \qquad \text{for all} \quad z \in \mathbf{C}, \tag{4.11a}$$

and with (4.8) it follows that

$$q(\nu_1; z) \geq c_0 + c_1 \tag{4.11b}$$

for all $z \in \mathbf{C}$ and all admissible sets S, but also for the extreme case $S = S(\mu_K)$. In the latter case we have

$$q(\nu_1; z) - (c_0 + c_1) \geq q(\omega_{S(\mu_K)}; z) - \log \operatorname{cap}(S(\mu_K)) \tag{4.12}$$

for quasi every $z \in S(\mu_K)$, where $\omega_{S(\mu_K)}$ is the equilibrium distribution on $S(\mu_K)$ and $S(\nu_1) \subseteq S(\mu_K)$. The right-hand side of (4.12) is the Green function $g_{\Omega_K}(z, \infty)$.

Since $c_0 + c_1 > -\infty$, it follows from (4.11b) that ν_1 is of finite energy. This implies that every set of capacity zero is of ν_1-measure zero (see Theorem 2.4 of [La]). Hence, (4.12) holds ν_1-almost everywhere, and by the principle of domination (Theorem 1.27 of [La]) it then follows that the inequality in (4.12) extends to all $z \in \mathbf{C}$.

From the maximum principle for harmonic functions we know that in (4.12) we have either a strict inequality or equality for all $z \in \Omega_K$. In the latter case the measure ν_1 is the equilibrium distribution $\omega_{S(\mu_K)}$. But this would imply that in (4.11a) and (4.11b) we have equality for quasi every $z \in S(\mu_K)$, which is not possible because of (4.5). Hence, in (4.12) we have a strict inequality in Ω_K. This proves (4.10) for the extreme case $S = S(\mu_K)$.

Formula (4.9) shows that c_1 decreases with a shrinking set S. The smallest value of c_1 is assumed if $S = S(\mu_K)$. Using this monotoicity and the continuity of the capacity from the outside, it can be shown that that (4.10) is already true for admissible sets S if they are only small enough.

After these preparations we can start with the construction of the monic polynomials U_n. From the definition of the measure ν_0 in (4.4), and from its decomposition in (4.6), it follows that for every $n \in N$ we can select a subset Z_{0n} of the zero sets $Z_n := Z(Q_n(\mu; \cdot))$ of the orthogonal polynomials $Q_n(\mu; \cdot)$ in such a way that

$$Z_{0n} \subseteq \mathbf{R} \setminus \mathring{S} \qquad \text{for all} \quad n \in N \tag{4.13a}$$

and

$$\frac{1}{n} \nu_{Z_{0n}} \xrightarrow{*} \nu_{01} \qquad \text{as} \quad n \to \infty, \quad n \in N. \tag{4.13b}$$

On the other hand it has been shown after Lemma 3.1 that for each $n \in N$ we can select a set Z_{1n} from S, which has exactly as many points as the Z_{0n}, and

$$\frac{1}{n} \nu_{Z_{1n}} \xrightarrow{*} \nu_{11} \qquad \text{as} \quad n \to \infty, \quad n \in N. \tag{4.14}$$

From (4.13a), (4.13b), and the fact that $S(\mu_K) \subseteq \mathring{S}$, it follows that

$$\lim_{n \to \infty, n \in N} \frac{1}{n} \log |Q(Z_{0n}; z)| = q(v_{01}; z) \qquad \text{uniformly for} \quad z \in S(\mu_K). \tag{4.15}$$

From the principle of descent of potential theory (Theorem 1.3 of [La]) and (4.14) we know that for any sequence of numbers $\{z_n\}$ with $z_n \to z_0 \in S(\mu_K)$ we have

$$\limsup_{n \to \infty, n \in N} \frac{1}{n} \log |Q(Z_{1n}; z_n)| \leqslant q(v_{11}; z_0).$$

The definition of v_{11} together with (4.8) implies that

$$q(v_{11}; z) = c_1 + q(v_{01}; z) \qquad \text{for all} \quad z \in S,$$

and this shows that $q(v_{11}; z)$ is continuous in $\mathring{S}$. Since $S(\mu_K) \subseteq \mathring{S}$, we therefore know that

$$\limsup_{n \to \infty, n \in N} \frac{1}{n} \log |Q(Z_{1n}; z)| \leqslant q(v_{11}; z) \qquad \text{uniformly for} \quad z \in S(\mu_K). \tag{4.16}$$

The polynomials $Q(Z_{0n}; \cdot)$ and $Q(Z_{1n}; \cdot)$ in (4.15) and (4.16) are monic polynomials with zero sets Z_{0n} and Z_{1n}, respectively. From the limits (4.15) and (4.16) together with (4.8) we derive that

$$\limsup_{n \to \infty, n \in N} \frac{1}{n} \log \left| \frac{Q(Z_{1n}; z)}{Q(Z_{0n}; z)} \right| \leqslant c_1 \qquad \text{uniformly for} \quad z \in S(\mu_K). \tag{4.17}$$

The monic polynomial U_n is now defined as

$$U_n(z) := Q(Z(Q_n(\mu; \cdot)) \backslash Z_{0n} \cup Z_{1n}; z) \equiv \frac{Q(Z_{1n}; z)}{Q(Z_{0n}; z)} Q_n(\mu; z) \tag{4.18}$$

for each $n \in N$. For the $L^2(\mu_K)$-norm of these polynomials we have the upper estimate

$$\begin{aligned} \|U_n\|_{L^2(\mu_K)} &\leqslant \sup_{x \in S(\mu_K)} \left| \frac{Q(Z_{1n}; z)}{Q(Z_{0n}; z)} \right| \|Q_n(\mu; \cdot)\|_{L^2(\mu_K)} \\ &\leqslant \sup_{x \in S(\mu_K)} \left| \frac{Q(Z_{1n}; z)}{Q(Z_{0n}; z)} \right| \|Q_n(\mu; \cdot)\|_{L^2(\mu)}. \end{aligned} \tag{4.19}$$

With the second limit in (4.4), the uniform upper estimate proved in (4.17), and the inequality (4.10), we deduce from (4.19) that

$$\limsup_{n \to \infty,\, n \in N} \frac{1}{n} \log \|U_n\|_{L^2(\mu_K)} \leqslant c_1 + c_0 < \log \operatorname{cap}(S(\mu_K)), \tag{4.20}$$

which proves the asymptotic inequality (4.3), and thereby the whole lemma. Q.E.D.

LEMMA 4.2. *Let $K := [a_1, a_2]$ be an interval. If there exists an infinite sequence of monic polynomials $\{U_n\} = \{U_n(z) = z^m + \cdots \in \Pi_n;\ m \leqslant n \in N \subseteq \mathbf{N}\}$ with real zeros, and if*

$$\operatorname{cap}\{z \in S(\mu_K);\ L_2(\mu_K, \{U_n\}; z) > 0\} > 0, \tag{4.21}$$

then we also have

$$\operatorname{cap}\{z \in S(\mu_K);\ L_0(\mu, \mathbf{N}; z) > 0\} > 0. \tag{4.22}$$

Remark. In a certain sense Lemma 4.2 is the converse of Lemma 4.1; we have only to set $U_n = Q_n(\mu; \cdot)$, $n \in N \subseteq \mathbf{N}$. But in Lemma 4.2 the set K is more special; it has to be an interval.

Proof. The basic idea of the proof is to combine the zero distributions of the elements of the two sequences $\{U_n\}$ and $\{P_n(\mu; \cdot)\}$ so that we can define a new sequence of polynomials with an asymptotic zero distribution like that of $\{U_n\}$ on a certain segment of $S(\mu_K)$ and like that of $\{P_n(\mu; \cdot)\}$ on the remaining parts of $S(\mu)$. The points, where these two subsets meet, demand special care. Generally speaking, the proof of the lemma requires rather detailed considerations, and is, unfortunately, rather long.

The proof is carried out indirectly: We assume that (4.21) holds true, while (4.22) is false, and show that this leads to a contradiction.

Without loss of generality, we can specialize the assumptions of the lemma in the following five aspects:

(i) By a simple linear transformation of the variable z, we can ensure that

$$\operatorname{cap}(I(\mu)) \leqslant 1. \tag{4.23}$$

(ii) By Lemma 3.6 we can assume that there exists an infinite subsequence $N_1 \subseteq N$ so that the limits

$$\frac{1}{n} \nu_{U_n} \xrightarrow{*} \nu_1 \qquad \text{and} \qquad \frac{1}{n} \log \|U_n\|_{L^2(\mu_K)} \to c_1 \in \mathbf{R} \cup \{-\infty\}$$

$$\text{for} \quad n \to \infty, \quad n \in N_1, \tag{4.24}$$

exist, and that (4.21) holds also true for the subsequence $\{U_n;\, n \in N_1\}$.

(iii) Without loss of generality we can assume that in (4.24) $c_1 > -\infty$. For otherwise we can modify the sequence $\{U_n; n \in N_1\}$ by formally increasing the index n (not the actual degree m) of the polynomials U_n. More precisely: Because $\operatorname{cap}(S(\mu_K)) > 0$ we have

$$\|U_n\|_{L^2(\mu_K)} > 0 \qquad \text{for all} \quad U_n, n \in N_1,$$

and therefore we can choose a constant c with $0 > c > -\infty$ and for every $n \in N_1$ an integer $m(n) > n$ such that

$$\frac{1}{m(n)} \log \|U_n\|_{L^2(\mu_K)} \to c \qquad \text{as} \quad n \to \infty, n \in N_1, \tag{4.25}$$

We define a new sequence $\{\tilde{U}_m\} = \{\tilde{U}_m; m \in N_2\}$ by $N_2 := \{m(n); n \in N_1\}$ and $\tilde{U}_m = U_n$ if $m = m(n)$. If $c_1 = -\infty$, then $n/m(n) \to 0$ for $n \in N_1$ and $m(n) \in N_2$, and therefore the limit measure ν_1 in (4.24) will be identical zero for the new sequence $\{\tilde{U}_m\}$. By Lemma 3.4 it then follows that

$$L_2(\mu_K, \{\tilde{U}_m\}; z) \equiv -c > 0, \tag{4.26}$$

which shows that (4.21) hold true for the new sequence and at the same time we have $c_1 = c > -\infty$ for this sequence.

(iv) Without loss of generality we can assume that

$$Z(U_n) \subseteq V \qquad \text{for all} \quad n \in N_1, \tag{4.27}$$

where $V \subseteq \mathbf{R}$ is a compact interval containing $S(\mu_K)$ in its interior. Indeed, it has been assumed that all zeros of U_n are real. If (4.27) is false, then by the balayage technique of Lemma 3.2, applied in the same way as in the proof of Lemma 4.1, we can construct a new sequence of monic polynomials so that the new polynomials have all their zeros in a compact interval V that contains $S(\mu_K)$ in its interior, and (4.21) holds true for the new sequence. From (4.27) it follows that $S(\nu_1) \subseteq V$.

(v) Without loss of generality, we can assume that

$$L_2(\mu_K, \{U_n\}; a_j) = -\infty \qquad \text{for} \quad j = 1, 2, \tag{4.28}$$

where a_1, a_2 are the end points of the interval K. Indeed, let b_1 be a positive constant, and consider the monic polynomials

$$V_n(b_1; z) := ((z - a_1)(z - a_2))^{[nb_1]} \qquad \text{for} \quad n \in \mathbf{N},$$

where $[nb_1]$ denotes the largest integer not greater than nb_1. Since

$$|(z - a_1)(z - z_2)|^{b_1} \to 1 \qquad \text{for} \quad b_1 \to 0$$

uniformly on compact subsets of $\mathbf{R}\setminus\{a_1, a_2\}$, it follows that for $b_1 > 0$ sufficiently small the sequence

$$\{\tilde{U}_{\tilde{n}} \in \Pi_{\tilde{n}};\, \tilde{U}_{\tilde{n}} = V_n(b_1; \cdot)\, U_n,\, \tilde{n} = n + [nb_1],\, n \in N_1\} \tag{4.29}$$

of monic polynomials satisfies (4.21) and (4.28) simultaneously. (It may be necessary to repeat here step (ii)).

After these preparations we can begin with the actual proof: The technical aim is to construct a sequence $\{V_n\}$ of monic polynomials so that the polynomials V_n have a $L^2(\mu)$-norm that asymptotically contradicts the minimality property (3.23) of the monic orthogonal polynomials $Q_n(\mu; \cdot)$. In a first step we shall construct the asymptotic zero distribution of the new sequence $\{V_n\}$.

In order to have a shorter notation we denote the upper limit function $L_2(\mu_K, \{U_n\}; \cdot)$ in (4.21) by h_1. Because of assumption (4.24) it then follows from Lemma 3.2 that

$$h_1(z) := L_2(\mu_K, \{U_n; n \in N_1\}\, z) \equiv q(\nu_1; z) - c_1\ , \tag{4.30}$$

and because of assumption (4.28) there exists a constant b_2 with $0 < b_2 < 1$ so that

$$\begin{aligned} &b_2 h_1(z) < g_\Omega(z, \infty) \\ &\text{for all} \quad z \in J(a_j) := \{z \in \mathbf{C}; \operatorname{Re}(z) = a_j\}, \quad j = 1, 2. \end{aligned} \tag{4.31}$$

Indeed, the inequality in (4.31) holds true in neighborhoods of the two points a_1 and a_2, and since both functions $h_1(z)$ and $g_\Omega(z, \infty)$ are continuous outside of $\mathbf{R}$, we can choose $b_2 > 0$ so small that the inequality in (4.31) holds true on the two lines $J(a_1)$ and $J(a_2)$. The constant b_2 will be kept fixed in the sequel.

By Helly's Selection Theorem, we can select an infinite subsequence of N_1, which we continue to denote by N_1, so that for the associated sequence

$$N_2 = N_2(b_2, N_1) := \left\{n_2 \in \mathbf{N};\, n_2 = \left[n \frac{1}{b_2}\right],\, n \in N_1\right\}, \tag{4.32}$$

the limits

$$\frac{1}{n}\, \nu_{Q_n(\mu;\, \cdot)} \xrightarrow{*} \nu_2$$

and (4.33)

$$\frac{1}{n} \log UQ_n(\mu; \cdot)\|_{L^2(\mu)} \to c_2 \in \mathbf{R} \cup \{-\infty\} \qquad \text{as} \quad n \to \infty, \quad n \in N_2$$

exist. In part (b) of Lemma 3.7 it has been shown that $S(\nu_2) \subseteq S(\mu)$. From (4.33) and the representation stated in Lemma 3.4 it follows that

$$h_2(z) := L_0(\mu, N_2; z) = q(\nu_2; z) - c_2 \qquad \text{for all} \quad z \in \mathbf{C}. \tag{4.34}$$

From Lemma 3.7, part (a), we know that

$$h_2(z) \geqslant g_\Omega(z, \infty) \qquad \text{for all} \quad z \in \mathbf{C}, \tag{4.35a}$$

and with the assumption that (4.22) is false it therefore follows that

$$h_2(z) = 0 \qquad \text{for} \quad z \text{ qu.e. on } S(\mu_K). \tag{4.35b}$$

Now, (4.21) and (4.35b) together imply that the set

$$\tilde{G} := \{z \in \mathbf{C} \backslash \mathbf{R};\, a_1 < \operatorname{Re}(z) < a_2,\, b_2 h_1(z) > {}_2(z)\} \tag{4.36}$$

is not empty. Since both functions h_1 and h_2 are continuous outside of $\mathbf{R}$, the set $\tilde{G}$ is open, and since $b_2 < 1$, it follows from (4.35a) that $\tilde{G}$ is bounded.

The set $\tilde{G}$ is symmetric with respect to $\mathbf{R}$. Let $\hat{G}$ be the union of two components of $\tilde{G}$ laying symmetric to $\mathbf{R}$, and let D be the unbounded component of the the complement of the closure of $\hat{G}$. From the maximum principle for harmonic functions it follows that the closure of $\hat{G}$ intersects $\mathbf{R}$. We define $G := \mathbf{C} \backslash \bar{D}$. The set G is bounded since $\tilde{G}$ is bounded, it is open, every component is simply connected by definition (actually, it will turn out that there is only one component), it is symmetric with respect to $\mathbf{R}$, and it is equal to the interior of its closure by definition. The boundary of G in $\mathbf{C} \backslash \mathbf{R}$ is contained in the boundary of $\tilde{G}$, and the set $J := \bar{G} \cap \mathbf{R}$ is an interval. In any case, J is connected. If J were not an interval, then it would be a single point, but this is impossible since J is of positive capacity, as we shall prove immediately. Since J is an interval, it follows that G is connected and therefore a domain.

We prove that

$$[a_3, a_4] := J \subseteq (a_1, a_2) \qquad \text{and} \qquad \operatorname{cap}(J \cap S(\mu)) > 0. \tag{4.37}$$

Indeed, the inclusion in (4.37) follows from (4.36). The proof of $\operatorname{cap}(J \cap S(\mu)) > 0$ needs some considerations: The function $d(z) := b_2 h_1(z) - h_2(z)$ is harmonic in $G \backslash \mathbf{R}$ and zero on $(\partial G) \backslash \mathbf{R}$. Hence, $(\partial G) \backslash \mathbf{R}$ consists piecewise of analytic arcs. The normal derivative of d on $(\partial G) \backslash \mathbf{R}$ directed toward G is positive on every analytic arc. Since the integral of the normal derivatives of a harmonic function along a closed curve is equal zero, it follows with the representations (4.30) and (4.34) that $b_2 \nu_1(J) -$

$\nu_2(J) < 0$, which is only possible if $\nu_2(J) > 0$. We note that $\partial G \cap \mathbf{R}$ consists of two points only.

Since h_2 is bounded from below (see (4.35a)), $\nu_2(J) > 0$ implies that $\operatorname{cap}(J \cap S(\nu_2)) > 0$ (see Theorem 2.4 of [La]). With $S(\nu_2) \subseteq S(\mu)$ this further implies that $\operatorname{cap}(J \cap S(\mu)) > 0$.

As already mentioned earlier, the basic idea of the proof is to copy the asymptotic zero distribution of the sequence $\{U_n\}$ on J, and the asymptotic zero distribution of $\{P_n(\mu; \cdot)\}$ on $I(\mu) \setminus J$ into a new sequence of monic polynomials. In order to find the appropriate asymptotic zero distribution for the new sequence, we consider the function

$$h_3(z) := \begin{cases} b_2 h_1(z) & \text{for } z \in G \\ h_2(z) & \text{for } z \in \mathbf{C} \setminus (G \cup \{a_3, a_4\}) \\ \max[h_2(z), b_2 h_1(z)] & \text{for } z \in \{a_3, a_4\}. \end{cases} \tag{4.38}$$

The function h_3 is subharmonic. Indeed, $(\partial G) \setminus \mathbf{R}$ is contained in $\partial \tilde{G}$, and the intersection

$$(\partial G \setminus \mathbf{R}) \cap \overline{(\partial \tilde{G} \setminus \partial G)}$$

consists of isolated points since at each of these points the complex derivative of $d = b_2 h_1 - h_2$ has a zero. We will call them critical points. Both functions $b_2 h_1$ and h_2 are harmonic in $\mathbf{C} \setminus \mathbf{R}$, and near $\partial G \setminus \{$critical points$\}$ the function h_3 is the maximum of these two functions. Hence, the subharmonicity of h_3 follows by a standard technique of potential theory (see Chapter I, Section 2 of [La]). In any case the subharmonicity of h_3 is critical only on ∂G. For an arbitrary point $z_0 \in \partial G \setminus \{a_3, a_4$, critical points$\}$ we will explicitly check the lower mean-value property of h_3. Let $\varepsilon > 0$ be sufficiently small, then

$$\{|z - z_0| \leqslant \varepsilon\} \cap G = \{|z - z_0| \leqslant \varepsilon\} \cap \tilde{G}$$

and from the definition of $\tilde{G}$ in (4.36) and the definition of h_3 in (4.38) it follows that

$$\begin{aligned} &\frac{1}{2\pi\varepsilon} \int_{|\zeta| = \varepsilon} h_3(z_0 + \zeta)\, ds_\zeta \\ &\quad = \frac{1}{2\pi\varepsilon} \left[\int_{G \cap \{|\zeta| = \varepsilon\}} b_2 h_1(z_0 + \zeta)\, ds_\zeta + \int_{\{|\zeta| = \varepsilon\} \setminus G} h_2(z_0 + \zeta)\, ds_\zeta \right] \\ &\quad \geqslant \frac{1}{2\pi\varepsilon} \int_{|\zeta| = \varepsilon} h_2(z_0 + \zeta)\, ds_\zeta \geqslant h_2(z_0) = h_3(z_0). \end{aligned} \tag{4.39}$$

The last equality follows from the fact that on $(\partial G)\backslash \mathbf{R}$, we have $h_3 = b_2 h_1 = h_2$. The isolated critical points on $(\partial G)\backslash \mathbf{R}$ cannot spoil the subharmonicity since both functions $b_2 h_1$ and h_2 are continuous in $\mathbf{C}\backslash \mathbf{R}$.

Since h_3 is subharmonic in $\mathbf{C}$, we have a representation as a logarithmic potential plus a constant (see Theorem 1.22 of [La]), i.e., there exists a probability measure ν_3 with

$$h_3(z) \equiv q(\nu_3; z) - c_2 \qquad \text{and} \qquad S(\nu_3) \subseteq S(\mu) \cup \partial G. \tag{4.40}$$

Since h_3 is identical with h_2 in a neighbourhood of infinity, ν_3 is a probability measure, and the constant in (4.40) is the same as that in (4.34).

We break down the measure ν_3 in three parts:

$$\nu_{31} := b_2 \nu_1|_J = \nu_3|_J, \qquad S(\nu_{31}) \subseteq J \cap S(\mu), \tag{4.41a}$$

$$\nu_{32} := \nu_2|_{S(\mu)\backslash J} = \nu_3|_{S(\mu)\backslash J}, \qquad S(\nu_{32}) \subseteq S(\mu)\backslash \mathrm{Int}(J), \tag{4.41b}$$

$$\nu_{30} := \nu_3 - (\nu_{31} + \nu_{32}),\ \nu_{30} \neq 0, \qquad S(\nu_{30}) \subseteq \partial G. \tag{4.41c}$$

All three measures are non-negative. It is not difficult to verify the equalities stated in (4.41a) to (4.41c). That the measure ν_{30} is not identical zero follows from the fact that $b_2 h_1(z) > h_2(z)$ for $z \in G$ near $\partial G \backslash \{a_3, a_4,$ critical points$\}$.

In the sequel we shall keep the two measures ν_{31} and ν_{32} fixed, while we will modify the measure ν_{30}. We apply the technique of balayage, and sweep the mass of ν_{30} out of the domain $\mathbf{C}\backslash I(\mu)$; however, this will be done in a special way: The mass is swept only partly (the greater part) to the interval $I = I(\mu)$, the remaining part is swept to the point infinity. We will explain the procedure in more detail: Let $\omega_z = \omega_{I,z}$ be the harmonic measure representing a point $z \in \bar{\mathbf{C}}\backslash I$ on the interval I (see Chapter IV, Section 1 of [La]). We consider the measure

$$\nu_{33} := \int \omega_{I,z}\, d\nu_{30}(z) - b_3 \omega_{I,\infty}, \tag{4.42}$$

where b_3 is a constant with $0 < b_3 < 1$, and compare the potentials of the two measures ν_{30} and ν_{33}. From the definition of the Green function it follows that for any fixed $x \in \mathbf{C}\backslash I$ we have

$$q(\omega_{I,x} - \delta_x; z) \equiv g_{\bar{\mathbf{C}}\backslash I}(z, x) - g_{\bar{\mathbf{C}}\backslash I}(x, \infty), \tag{4.43a}$$

and from the representation of the Green function by the equilibrium distribution (see Theorem 2.6 of [La]) we further know that

$$q(\omega_{I,\infty}; z) \equiv g_{\bar{\mathbf{C}}\backslash I}(z, \infty) + \log \mathrm{cap}(I). \tag{4.43b}$$

The harmonic measure $\omega_{I,\infty}$ is the equilibrium distribution of I. The definition (4.42) of the measure ν_{33} together with (4.43a) and (4.43) yields

$$q(\nu_{33}-\nu_{30};z)\equiv\int q(\omega_{I,x}-\delta_x;z)\,d\nu_{30}(x)-b_3q(\omega_{I,\infty};z)$$

$$\equiv\int [g_{\bar{\mathbf{C}}\setminus I}(z,x)-g_{\bar{\mathbf{C}}\setminus I}(x,\infty)]\,d\nu_{30}(x)$$

$$-b_3[g_{\bar{\mathbf{C}}\setminus I}(z,\infty)+\log\operatorname{cap}(1)]. \tag{4.43c}$$

This shows that the measure ν_{33} is positive for b_3 sufficiently small. Hence, the measure ν_{33} results from sweeping out a proportion $(1-b_3)$ of the measure ν_{30} onto the interval I, and sweeping the remaining proportion b_3 of ν_{30} to infinity.

Since the interval I is a regular set, it follows that the left-hand side of (4.43c) is constant on I. We have

$$q(\nu_{33}-\nu_{30};z)=-\int g_{\bar{\mathbf{C}}\setminus I}(x,\infty)\,d\nu_{30}(x)-b_3\log\operatorname{cap}(I)=:c_3<0 \tag{4.44a}$$

for all $z\in I$, where the inequality in (4.44a) holds true only if the constant $b_3>0$ has been chosen sufficiently small, which we will assume in the sequel.

From (4.42) it then follows that

$$\nu_{33}(\mathbf{C})=\nu_{30}(\mathbf{C})-b_3<\nu_{30}(\mathbf{C}) \qquad \text{and} \qquad S(\nu_{33})\subseteq I. \tag{4.44b}$$

Up to a small modification, which will be introduced later in steps (iv) and (v) below, the measure

$$\nu_4:=\nu_{31}+\nu_{32}+\nu_{33} \tag{4.45a}$$

is the asymptotic zero distribution we were looking for. We have

$$\nu_4(\mathbf{C})=1-b_3<1. \tag{4.45b}$$

From (4.40) and (4.44a) it follows that

$$h_3(z)=q(\nu_4;z)-c_4 \qquad \text{for all } z\in I \qquad \text{with} \quad c_4:=c_2-c_3>c_2. \tag{4.46}$$

We now come to the final stage of the proof, the construction of a sequence $\{V_n; n\in N_2\}$ of monic polynomials. The zero set of each of the polynomials V_n will be selected in the form of five separate subsets, which are denoted by $Z_{j,n}$, $j=1,\ldots,5$, $n\in N_2$.

(i) Let $Z_{1,n}$, $n \in N_2$, be the set of all zeros of the polynomial $U_{[nb_2]}$ on J. Then from (4.27), from the assumed limits in (4.24), from the definition of the sequence N_2 in (4.32), and from (4.4a), it follows that

$$\frac{1}{n} \nu_{Z_{1,n}} \xrightarrow{*} \nu_{31} \qquad \text{for} \quad n \to \infty, \quad n \in N_2. \tag{4.47}$$

We note that $\nu_1(\{a_j\}) = \nu_3(\{a_j\}) = 0$ for $j = 3, 4$, since otherwise h_1 and therefore also h_3 would not be bounded from below, which has to be the case because of (4.35a) and Definition (4.38).

(ii) Let $Z_{2,n}$, $n \in N_2$, be the set of all zeros of the polynomial $Q_n(\mu; \cdot)$ on $I \backslash J$. From the limits in (4.33) and from Definition (4.41b), it follows that

$$\frac{1}{n} \nu_{Z_{2,n}} \xrightarrow{*} \nu_{32} \qquad \text{for} \quad n \to \infty, \quad n \in N_2. \tag{4.48}$$

(iii) Let $Z_{3,n}$, $n \in N_2$, be a set of $[n \, \|\nu_{33}\|]$ points from the interval I so that

$$\frac{1}{n} \nu_{Z_{3,n}} \xrightarrow{*} \nu_{33} \qquad \text{for} \quad n \to \infty, \quad n \in N_2. \tag{4.49}$$

By Lemma 3.1 such a selection can be made for each $n \in N_2$.

(iv) Let b_4 be a constant such that

$$b_4 \log |(z - a_3)(z - a_4)| \leqslant \frac{-c_3}{2} \qquad \text{for all } z \in I \qquad \text{and} \qquad 0 < 2b_4 < b_3. \tag{4.50}$$

Then the set $Z_{4,n}$, $n \in N_2$, is defined as $[nb_4]$ repetitions of the two points a_3 and a_4. The sets $Z_{4,n}$ are instrumental in eliminating problems at the end points of the interval J.

(v) Let be $n_j := \text{card}(Z_{j,n})$, $j = 1, ..., 4$, $n \in N_2$. For $n \in N_2$ sufficiently large, we have $n_5 := n - n_1 - \cdots - n_4 > 0$. For each $n \in N_2$ with $n_5 > 0$, we select a set $Z_{5,n}$ of n_5 point from the interval I in such a way that

$$\frac{1}{n_5} \nu_{Z_{5,n}} \xrightarrow{*} \omega_I \qquad \text{for} \quad n \to \infty, \quad n \in N_2, \tag{4.51}$$

where ω_I is the equilibrium distribution on I. Because of assumption (4.23), we have

$$\limsup_{n \to \infty,\, n \in N_2} \frac{1}{n} q(\nu_{z_{5,n}}; z) \leqslant 0 \qquad \text{uniformly for } z \in I. \tag{4.52}$$

After these five separate definitions, we define

$$Z_n := Z_{1,n} \cup \cdots \cup Z_{5,n}, \tag{4.53a}$$

and

$$V_n(z) := Q_n(Z_n; z) \qquad \text{for} \quad n \in N_2. \tag{4.53b}$$

Hence, every V_n is a monic polynomial of degree n. From the limits in (4.47), (4.48), (4.49), and (4.51) it follows that

$$\begin{aligned} \frac{1}{n} \nu_{Z_n} &\xrightarrow{*} \nu_{31} + \nu_{32} + \nu_{33} + b_4(\delta_{a_3} + \delta_{a_4}) + b_5 \omega_I \\ &= \nu_4 + b_4(\delta_{a_3} + \delta_{a_4}) + b_5 \omega_I \qquad \text{for} \quad n \to \infty, \quad n \in N_2, \end{aligned} \tag{4.53c}$$

where $b_5 = b_3 - 2b_4 > 0$. The equality in (4.53c) follows from the definition of the measure ν_4 in (4.45a). Using Lemma 3.1, the identities (4.40), (4.41c), (4.44a), (4.45a), and the asymptotic estimates (4.50) and (4.52), we deduce from (4.53c) that for any sequence of points $\{z_n\}$ with $z_n \to z_0 \in \mathbf{C}$ as $n \to \infty$, we have

$$\limsup_{n \to \infty,\, n \in N_2} \frac{1}{n} q(\nu_{Z_n}; z_n) \leqslant h_3(z_0) + c_4 - \frac{1}{2} c_3 = h_3(z_0) + c_2 + \frac{1}{2} c_3. \tag{4.53d}$$

For the $L^2(\mu)$-norm of the polynomials V_n we will now derive an asymptotic estimate. But first we prove some auxiliary results: For $\varepsilon > 0$ and $n \in N_2$ sufficiently large, we have

$$\begin{aligned} |V_n(z)| &= |U_{[nb_2]}(z)| \left| \frac{Q(Z_n \backslash Z_{1,n}; z)}{Q(Z(U_{[nb_2]}) \backslash Z_{1,n}; z)} \right| \\ &\leqslant |U_{[nb_2]}(z)| \, e^{n(c + (1/2)c_3 - b_2 c_1 + \varepsilon)} \end{aligned} \tag{4.54}$$

for all $z \in J$, and

$$|V_n(z)| = |Q_n(\mu; z)| \left| \frac{Q(Z_n \backslash Z_{2,n}; z)}{Q(Z(Q_n(\mu; \cdot)) \backslash Z_{2,n}; z)} \right| \leqslant |Q_n(\mu; z)| \, e^{n((1/2)c_3 + \varepsilon)} \tag{4.55}$$

for all $z \in I \backslash J$. Indeed, by the definition of the sets $Z_n, Z_{1,n}, ..., Z_{5,n}$ we have the identity

$$\nu_{Z_{2,n} \cup \cdots \cup Z_{5,n}} - \nu_{Z(U_{[nb_n]}) \backslash Z_{1,n}} = \nu_{Z_n} - \nu_{Z(U_{[nb_2]})}. \tag{4.56}$$

and we see that the measure on both sides of (4.56) has no negative mass on the interval J, which implies that the rational function in the second term of (4.54) has no poles on the interval J, and we can therefore expect that there exists an upper estimate. In order to show this we consider the following asymptotic estimate: For any sequence of points $\{z_n\}$ with $z_n \to z_0 \in J$ as $n \to \infty$, we have

$$\begin{aligned}
&\limsup_{n \to \infty, n \in N_2} \frac{1}{n} \log \left| \frac{Q(Z_n \setminus Z_{1,n}; z_n)}{Q(Z(U_{[nb_2]}) \setminus Z_{1,n}; z_n)} \right| \\
&= \limsup_{n \to \infty, n \in N_2} \frac{1}{n} q(v_{Z_n} - v_{Z(U_{[nb_2]})}; z_n) \\
&\leqslant q(v_4 + b_4(\delta_{a_3} + \delta_{a_4}) + b_5 \omega_{I,\infty} - b_2 v_1; z_0) \\
&\leqslant q(v_3 - b_2 v_1; z_0) + \frac{c_3}{2} \\
&= h_3(z_0) + c_2 - b_2[h_1(z_0) + c_1] + \frac{c_3}{2} \leqslant c_2 - b_2 c_1 + \frac{c_3}{2}. \qquad (4.57)
\end{aligned}$$

Indeed, the identity of the two limits in (4.57) follows from identity (4.56). Because of the limits (4.24) and (4.53c) and because of the fact that the measure on both sides of (4.56) has no negative mass on J, it follows from Lemma 3.1 that the first inequality in (4.57) holds true for every $z_0 \in \text{Int}(J)$. That this inequality holds also true at the two end points $z_0 = a_3$ and $z_0 = a_4$ of the interval J is a consequence of the selection of the sets $Z_{4,n}$, which ensures that the first two terms in (4.57) are close to minus infinity in neighborhoods of the two points a_3 and a_4. The second inequality in (4.57) follows from the identities (4.41c), (4.45a), (4.44a), and the inequalities (4.50) and (4.52). The next equality in (4.57) follows from the identities (4.30) and (4.40). The last inequality in (4.57) follows from the fact that by (4.38) and the definition of the domain G we know that $h_3(z) \leqslant b_2 h_1(z)$ for all $z \in J$, and equality holds true for all $z \in \text{Int}(J)$. From (4.57) then follows the estimate (4.54).

In order to prove (4.55), we proceed in a similar way as in the verification of inequality (4.54). First, we consider the identity

$$v_{Z_{1,n} \cup Z_{3,n} \cup Z_{4,n} \cup Z_{5,n}} - v_{Z(Q_n(\mu;\cdot)) \setminus Z_{2,n}} = v_{Z_n} - v_{Z(Q_n(\mu;\cdot))}. \qquad (4.58)$$

This identity follows like (4.56) from the definition of the sets $Z_n, Z_{1,n}, ..., Z_{5,n}$. We see that the measure on both sides of (4.58) has no negative mass on the set $I \setminus J$. Like in (4.57), we prove that for any sequence of points $\{z_n\}$ with $z_n \to z_0 \in I \setminus J$ as $n \to \infty$, we have

$$\begin{aligned}
\limsup_{n\to\infty,\, n\in N_2} \frac{1}{n}\log\left|\frac{Q(Z_n\backslash Z_{2,n}; z_n)}{Q(Z(Q_n(\mu;\cdot))\backslash Z_{2,n}; z_n)}\right| \\
&= \limsup_{n\to\infty,\, n\in N_2} \frac{1}{n} q(\nu_{Z_n} - \nu_{Z(Q_n(\mu;\cdot))}; z_n) \\
&\leqslant q(\nu_4 + b_4(\delta_{a_3} + \delta_{a_4}) + b_2\omega_{I,\infty} - \nu_2; z_0) \\
&\leqslant q(\nu_3 - \nu_2; z_0) + \frac{c_3}{2} \\
&= h_3(z_0) + c_2 - [h_2(z_0) + c_2] + \frac{c_3}{2} = \frac{c_3}{2}.
\end{aligned} \tag{4.59}$$

Now, the identity of the two limits in (4.59) follows from identity (4.58). That the first inequality in (4.59) holds true for every $z_0 \in I\backslash \mathrm{Int}(J)$ follows by Lemma 3.1 from the fact that the measure on both sides of (4.58) has no negative mass on $I\backslash J$ and from the limits (4.33) and (4.53c). That this inequality holds also for the two end points of J is again, like in (4.57), a consequence of the special selection of the sets $Z_{4,n}$. The second inequality and the last two equalities in (4.59) follow in exactly the same way as the corresponding relations in (4.57): The second inequality follow from the identities (4.41c), (4.45a), (4.44a) and the inequalities (4.50) and (4.52), the next equality follows from the identities (4.34) and (4.40), and the last equality from (4.38). From (4.59) then follows the estimate (4.55).

From (4.54) and (4.55) we deduce that for $\varepsilon > 0$ and $n \in N_2$ sufficiently large the upper estimate

$$\begin{aligned}
\int |V_n|^2\, d\mu &\leqslant e^{2n(c_2 + (1/2)c_3 - b_2c_1 + \varepsilon)} \int_J |U_{[nb_2]}|^2\, d\mu \\
&\quad + e^{2n((1/2)c_3 + \varepsilon)} \int_{I\backslash J} |Q_n(\mu;\cdot)|^2\, d\mu \\
&\leqslant e^{2n(c_2 + (1/2)c_3 - b_2c_1 + \varepsilon)} e^{2[nb_2](c_1 + \varepsilon)} + e^{2n((1/2)c_3 + \varepsilon} e^{2n(c_2 + \varepsilon)} \\
&\leqslant 2e^{2n(c_2 + (1/2)c_3 + 4\varepsilon)}.
\end{aligned} \tag{4.60}$$

The second inequality is a consequence of the second limits in (4.24) and (4.33) together with (4.32). Since $\varepsilon > 0$ was arbitrary and $c_3 < 0$, it follows from (4.60) with the second limit in (4.24) that

$$\|V_n\|_{L^2(\mu)} \leqslant 2e^{(n/3)c_3} \|Q_n(\mu;\cdot)\|_{L^2(\mu)} < \|Q_n(\mu;\cdot)\|_{L^2(\mu)} \tag{4.61}$$

for $n \in N_2$ sufficiently large. This inequality contradicts the minimality property (3.23) of the orthogonal monic polynomials $Q_n(\mu;\cdot)$. Hence, the assumption that (4.22) is false has been disproved, and Lemma 4.2 is proved. Q.E.D.

5. Proof of Theorem 2.1

It has been mentioned in Remark 3 to Theorem 2.1 that there are two groups of assertions in Theorem 2.1: The assertions (b) and (c) and the assertions (a), (d), and (e). The first group follows from the second one without the additional assumption (2.1) of Theorem 2.1, while for the proof of the reverse direction assumption (2.1) is necessary. The proof of Theorem 2.1 wil be organized in such a way that this structure becomes apparent.

Lemma 5.1. *If the asymptotic estimate* (2.4) *in assertion* (d) *of Theorem* 2.1 *is false for a sequence* $\{U_n; n \in N \subseteq \mathbf{N}\}$ *of polynomials, then there exists also an infinite sequence* $\{V_n; n \in N\}$ *of monic polynomials with real zeros, for which the estimate* (2.4) *is again false.*

Proof. The sequence of polynomials $\{U_n; n \in N\}$ will be transformed in two stages in a sequence $\{V_n; n \in N \subseteq \mathbf{N}\}$ of monic polynomials with real zeros. In the first stage balayage is used in a similar way as in Lemma 4.1.

Without loss of generality we may assume that the polynomials U_n are monic since the expression on the left-hand side of (2.4) is invariant under multiplication by a non-zero constant.

If the asymptotic estimate (2.4) does not hold true, then there exist $x_0 \in \mathbf{C}$ with $x_n \in \mathbf{C}$, $n \in N$, $x_n \to x_0$ as $n \to \infty$, $n \in N$, and

$$\limsup_{n \to \infty,\, n \in N} \frac{1}{n} \log \left| \frac{U_n(x_n)}{\|U_n\|_{L^2(\mu_K)}} \right| > g_{\Omega_K}(x_0, \infty). \tag{5.1}$$

By the definition of the limit function L_2 in Definition 3.3 there exist $\varepsilon > 0$ such that

$$L_2(\mu_K, \{U_n; n \in N\}; x_0) > g_{\Omega_K}(x_0, \infty) + \varepsilon. \tag{5.2}$$

Let us first assume that the zeros of all polynomials U_n are contained in a bounded set. By Helly's Selction Theorem we can select an infinite subsequence of N, which we continue to denote by N, such that the two limits in (3.10) exist; i.e.,

$$\frac{1}{n} \nu_{U_n} \xrightarrow{*} \nu_1 \qquad \text{and} \qquad \frac{1}{n} \log \|U_n\|_{L^2(\mu_K)} \to c_1 \in \mathbf{R} \cup \{-\infty\}$$
$$\text{as} \quad n \to \infty, \quad n \in N. \tag{5.3}$$

Of course, (5.2) holds also true for this subsequence. From Lemma 3.4 it then follows that both sides in (5.2) are logarithmic potentials plus a constant, and therefore the inequality (5.2) holds true in a neighborhood of x_0 in Cartan's fine topology, i.e., in a classical neighborhood minus a set that is thin near x_0 (see Section 3 of Chapter V of [La]). Hence, there exists $x \in \mathbf{C}$ with $\operatorname{Im}(x) \neq 0$ such that (5.2) holds true if we there replace x_0 by x. In order to keep the notation simple we assume that we have $\operatorname{Im}(x_0) =: y \neq 0$ already for the original point x_0.

Let $\delta > 0$ be such that

$$\left| \log \frac{|y|}{|y| + \delta} \right| < \frac{\varepsilon}{3}, \tag{5.4}$$

and assume further that V is a regular, compact set with

$$S(\mu_K) \subseteq \mathring{V}, \qquad V \subseteq \{z;\, |\operatorname{Im}(z)| \leqslant \delta\}, \tag{5.5}$$

and $x_0 \notin V$. As in the proof of Lemma 4.1 we can, by balayage, sweep all zeros of each polynomial U_n, $n \in N$, from outside V onto the boundary ∂V, and approximate then the balayage measure by discrete measures (see Lemma 3.1). This allows us to prove that there exists a sequence $\{W_n;\, n \in N\}$ of monic polynomials with $\deg(U_n) = \deg(W_n)$, all the zeros of W_n are contained in V, and there exists a constant c_0 such that

$$\lim_{n \to \infty,\, n \in N} \frac{1}{n} \log \left| \frac{U_n(z)}{W_n(z)} \right| = c_0 \tag{5.6a}$$

locally uniformly for $z \in \mathring{V}$, and

$$\limsup_{n \to \infty,\, n \in N} \frac{1}{n} \log \left| \frac{U_n(z)}{W_n(z)} \right| \leqslant c_0 \tag{5.6b}$$

locally uniformly for $z \in \mathbf{C} \setminus V$. From (5.6a) and (5.6b) and Definition 3.3 it follows that

$$L_2(\mu_K, \{W_n\}; x_0) \geqslant L_2(\mu_K, \{U_n\}; x_0). \tag{5.7}$$

For every $n \in N$ we now move the non-real zeros of the polynomial W_n perpendicular onto $\mathbf{R}$. The resulting new polynomial is denoted by V_n. Elementary calculations show that

$$|V_n(z)| \leqslant |W_n(z)| \qquad \text{for all} \quad z \in \mathbf{R} \quad \text{and} \quad n \in N. \tag{5.8}$$

Because of the assumptions made in (5.4) and (5.5), and since $\operatorname{Im}(x_0) = y$, we further have

$$\frac{1}{n} \log |V_n(z)| \geqslant \frac{1}{n} \log |W_n(z)| - \frac{\varepsilon}{3} \tag{5.9}$$

for all $n \in N$ and all z in a neighborhood of x_0. Hence, by (5.2) and (5.7) we have

$$L_2(\mu_K, \{V_n; n \in N\}; x_0) \geqslant g_{\Omega_K}(x_0, \infty) + \frac{\varepsilon}{3}, \tag{5.10}$$

which proves that assertion (d) of Theorem 2.1 is false for the sequence $\{V_n; n \in N\}$, and all zeros of the polynomials V_n are real.

It has been assumed that the zeros of all polynomials U_n are contained in a bounded set. As in the proof of Lemma 3.5 we will show that this additional assumption is not really necessary. Let $R > 0$ be so large that $S(\mu_K)$ and x_0 are contained in $\{|z| < R\}$ and factor each polynomial U_n in a product $U_{n,1} U_{n,2}$ of two monic polynomials $U_{n,1}$ and $U_{n,2}$ such that $U_{n,1}$ has all its zeros in $\{|z| \leqslant kR\}$ and $U_{n,2}$ has all its zeros in the complement. If the constant $k > 1$ is chosen large enough, then it follows from (3.17) that (5.2) holds also for the sequence $\{U_{n,1}\}$. This sequence is then used, instead of the original sequence $\{U_n\}$, for the construction of $\{V_n\}$. Q.E.D.

After these preparations we come to the main topic of the present section, the proof of Theorem 2.1.

Proof of Theorem 2.1. We first show the equivalence of the two assertions (b) and (c), then the equivalence of the three assertions (a), (d), and (e), and after that the equivalence of the two groups. Assumption (2.1) is used only in the proof of the implication (b) $\Rightarrow$ (e).

(b) $\Leftrightarrow$ (c): Assertion (c) follows from assertion (b), the lower estimate (1.6) in Lemma 1.1, and the fact that $g_{\Omega_K}(z, \infty) = 0$ qu.e. on $S(\mu_K)$ (see Theorem 2.6 of [La]).

On the other hand, assertion (c) implies that assumption (3.13) of Lemma 3.5 is satisfied, and assertion (b) then follows from the asymptotic inequality (3.14).

(d) $\Leftrightarrow$ (e): Assertion (e) is a special case of assertion (d) since $g_{\Omega_K}(z, \infty) = 0$ quasi everywhere on $S(\mu_K)$.

The other direction (e) $\Rightarrow$ (d) follows from Lemma 3.5 in a similar way as the implication (c) $\Rightarrow$ (b). In more detail: From Remark 1 to Definition 3.3 we know that we can assume without los of generality that the polynomials U_n in assertion (d) and (e) are monic. From the asymptotic inequality (2.5) in assertion (e) and from the definition of $\tilde{L}_2(\mu, \{U_n\}; z)$ in (3.7) it then follows by Lemma 3.5 that (3.14) holds true. Together with Remark 3 to Definition 3.3 this implies assertion (d).

(d) $\Rightarrow$ (a): It follows from assertion (d) that

$$\limsup_{n \to \infty} |P_n(\mu_K; z)|^{1/n} \leqslant e^{g_{\Omega_K}(z, \infty)} \tag{5.11}$$

for all $z \in \mathbf{C}$. Together with the lower asymptotic estimate (1.4) of Lemma 1.1, the asymptotic inequality (5.11) implies (1.7) of Lemma 1.2 if we there replace the measure μ by μ_K and the outer domain Ω by Ω_K. Hence, we have proved that the sequence $\{P_n(\mu_K; \cdot);\ n \in \mathbf{N}\}$ has regular asymptotic behavior.

(a) $\Rightarrow$ (d): Let us assume that assertion (d) is false. From Lemma 5.1 it then follows that there exists an infinite sequence $\{U_n\} = \{U_n \in \Pi_n;\ n \in N \subseteq \mathbf{N}\}$ of monic polynomials with real zeros such that the asymptotic inequality (2.4) does not hold true. From Remark 3 to Definition 3.3 we know that for this sequence the conclusion (3.14) of Lemma 3.5 is false if we replace μ by μ_K in the lemma. Hence, also (3.13) has to be false, which implies that

$$\operatorname{cap}\{z \in S(\mu_K);\ L_2(\mu_K, \{U_n\}; z) > 0\} > 0. \tag{5.12}$$

If we take $\mu = \mu_K$ in Lemma 4.2 and choose $a_1,\ a_2 \in \mathbf{R}$ such that $S(\mu_K) \subseteq [a_1, a_2]$, then it follows from (4.22) that

$$\operatorname{cap}\{z \in S(\mu_K);\ L_0(\mu_K, \mathbf{N}; z) > 0\} > 0. \tag{5.13}$$

Because of assertion (c) in Lemma 3.8, the inequality (5.13) contradicts regular asymptotic behavior of the sequence $\{P_n(\mu_K; \cdot)\}$, which proves the implication (a) $\Rightarrow$ (d).

We note that in the deduction of (5.13) from (5.12) we have not used the full power of Lemma 4.2 since we have assumed $S(\mu_K) \subseteq [a_1, a_2]$. The situation is different in the proof of the implication (c) $\Rightarrow$ (d) below.

Up to now, we have proved the equivalence of the assertions within the two groups $\{$(b), (c)$\}$ and $\{$(a), (d), (e)$\}$. We finish the proof by showing the equivalence of the two groups.

(a) $\Rightarrow$ (c): The implication will be proved indirectly. Let us assume that assertion (c) is false. From this assumption it follows that

$$\operatorname{cap}\{z \in S(\mu_K);\ L_0(\mu, \mathbf{N}; z) > 0\} > 0 \tag{5.14}$$

(see Definition 3.3). In Lemma 4.1 it has been shown that (5.14) implies (5.13). In the same way as after (5.13), we deduce with the help of assertion (d) of Lemma 3.8 that inequality (5.13) contradicts regular asymptotic behavior of the sequence $\{P_n(\mu_K; \cdot),\ n \in \mathbf{N}\}$, which proves the implication (a) $\Rightarrow$ (c).

(c) $\Rightarrow$ (d): Only here we use assumption (2.1) of Theorem 2.1; i.e., we now assume that

$$\operatorname{cap}(K \cap S(\mu)) = \operatorname{cap}(\mathring{K} \cap S(\mu)). \tag{5.15}$$

The implication (c) $\Rightarrow$ (d) will be proved indirectly. Let us assume that assertion (d) is false, while assertion (c) holds true. Then, as in the proof of the implication (a) $\Rightarrow$ (d), we can deduce from the assumption that assertion (d) is false that there exists an infinite sequence $\{U_n \in \Pi_n;\, n \in N \subseteq \mathbf{N}\}$ of monic polynomials with real zeros such that (5.12) holds true.

Set

$$S := \{z \in \mathring{K} \cap S(\mu);\, L_2(\mu_K, \{U_n\}; z) > 0\}. \tag{5.16}$$

We will show that (5.12) implies $\operatorname{cap}(S) > 0$. From (5.12) and (5.15) it follows that $\operatorname{cap}(\mathring{K} \cap S(\mu)) > 0$. There exist compact sets $V \subseteq \mathring{K} \cap S(\mu)$ with $\operatorname{cap}(S(\mu_V)) > 0$. Let V be such a set, and assume that

$$L_2(\mu_K, \{U_n\}; z) \leqslant 0 \tag{5.17}$$

for quasi every $z \in V$. Then it follows from Lemma 3.5 that

$$L_2(\mu_K, \{U_n\}; z) \leqslant g_{\bar{\mathbf{C}} \setminus V}(z, \infty) \qquad \text{for all} \quad z \in \mathbf{C}, \tag{5.18}$$

where in Lemma 3.5 we have to replace μ by μ_K, and μ_K by μ_V. Since $\mathring{K} \cap S(\mu)$ can be exhausted by sets V of the considered type, it follows that

$$L_2(\mu_K, \{U_n\}; z) \leqslant g_{\bar{\mathbf{C}} \setminus (\mathring{K} \cap S(\mu))}(z, \infty) = g_{\bar{\mathbf{C}} \setminus S(\mu_K)}(z, \infty) \tag{5.19}$$

for all $z \in \mathbf{C}$. The equality in (5.19) is a consequence of (2.8) and (5.15). The inequality (5.19) contradicts (5.12). Hence, (5.17) is false for some sets V, and this implies that $\operatorname{cap}(S) > 0$.

The set $\mathring{K}$ is the union of at most countably many open intervals. Since a countably infinite union of sets of capacity zero is again a set of capacity zero (see the corollary to Theorem 2.2 of [La]), at least one of these intervals, which we will denote by (a_1, a_2), satisfies

$$\operatorname{cap}(S \cap (a_1, a_2)) > 0. \tag{5.20}$$

We set $K_1 := [a_1, a_2]$. It is easy to see that (5.16) and the inequality (5.20) imply

$$\operatorname{cap}\{z \in S(\mu_{K_1});\, L_2(\mu_{K_1}, \{U_n\}; z) > 0\} > 0. \tag{5.21}$$

If we now apply Lemma 4.2 to the sequence $\{U_n\}$ and replace K by K_1 in Lemma 4.2, then it follows from this lemma and (5.21) that

$$\operatorname{cap}\{z \in S(\mu_{K_1});\, L_0(\mu, \mathbf{N}; z) > 0\} > 0. \tag{5.22}$$

The inequality (5.22) contradicts assertion (c). Indeed, from the asymptotic inequality (2.3) in assertion (c), from Definition 3.3, and from Lemma 3.5, it follows that

$$L_0(\mu, \mathbf{N}; z) \leqslant 0 \tag{5.23}$$

for z quasi everywhere on $S(\mu_K)$. Since $S(\mu_{K_2}) \subseteq S(\mu_K)$, inequality (5.23) contradicts (5.22). Hence, we have proved the implication (c) $\Rightarrow$ (d), and this completes the proof of Theorem 2.1. Q.E.D.

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